A STATISTICAL INVESTIGATION OF A DIVISOR-SUM FUNCTION

IVAN AIDUN AND LOLA THOMPSON

Abstract. The sum of proper divisors function $s(n)$ has been studied for more than 2000 years. In this paper we study statistical properties of the related function $S_s(n) := \sum_{d|n} s(d)$. This function arises from a generalization of the practical numbers. We prove that $S_s(n)/n$ has a continuous asymptotic distribution function, and that its values are dense in the interval $[0, \infty)$. We also establish mean value computations for $S_s(n)$ and $S_s(n)/n$, and provide uniform bounds for the higher order moments of $S_s(n)/n$. The main novelty in this paper is that we highlight a new method of Lebowitz-Lockard and Pollack that is useful for showing that certain functions have a continuous distribution function where classical methods sometimes fail.

1. Introduction

When faced with an erratic function, it is natural to want to understand its behavior: what are its extreme values? How does it behave on average? How does it behave "typically"? These types of questions have been studied for a panoply of arithmetic functions since the early part of the 20^{th} century. Going a step further, one can regard an arithmetic function f as a random variable on the discrete interval of integers in $[1, N]$, endowed with the uniform distribution, and apply tools from probability theory in order to study these functions.

The aim of this paper is to answer a number of probabilistic questions concerning a new function that we call $S_s(n)$. This function, which we define at the end of this section, has connections to the sum of proper divisors function, $s(n)$, and the practical numbers. In what follows, we provide background on $s(n)$ and the practical numbers, motivating the study of $S_s(n)$.

1.1. The function $s(n)$.

Let $s(n)$ denote the sum over all positive proper divisors of n, i.e.,

$$s(n) = \sum_{\substack{d|n \\ d<n}} d.$$

We may write $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the usual sum-of-divisors function. Note that $s(n)$ is neither additive nor multiplicative.

The function $s(n)$ has an ancient history, having been considered by the Pythagoreans. Pomerance [10] goes so far as to refer to $s(n)$ as "the first function". Despite being studied for over 2000 years, surprisingly little is known about $s(n)$ today.

The Pythagoreans were interested in classifying integers according to whether they satisfy $s(n) < n$, $s(n) > n$, or $s(n) = n$. Such integers are called *deficient*, *abundant*, or *perfect* numbers, respectively. It is natural to wonder how many of each of these numbers there are. There are known to be infinitely many abundant numbers; indeed, every multiple of 6 greater than 6 itself is abundant. It is not currently known if there are infinitely many perfect numbers. Euclid devised a method for generating perfect numbers, showing that a number

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of the form $2^{p-1}(2^p-1)$ is perfect if 2^p-1 is prime. Euler went on to prove that all even perfect numbers must have this form. No odd perfect numbers are known.

Because of the very restrictive form perfect numbers can take, it is not surprising that they are rare. It was first shown by Davenport [8] that the deficient, abundant, and perfect numbers all have asymptotic densities, and that the density of the perfect numbers is 0.

1.2. **The f-practical numbers.** The *practical numbers*, introduced by Srinivasan in [18], are positive integers n such that every number between 1 and n can be represented as a sum of distinct divisors of n.

There is a long history of studying the distribution of the practical numbers. Erdős [4] was the first to assert that the practical numbers have density 0. Complete criteria for a number to be practical were given by Stewart [19] and Sierpiński [17]. Let $P(x)$ denote the number of practical numbers less than or equal to x. The first bound on $P(x)$ was given by Hausman and Shapiro [5], who showed that $P(x) \leq x/(\log x)^{\beta+o(1)}$ for some constant $\beta > 0$ (though their original value of β was incorrect [9]).

Tenenbaum [20] established the sharper result $P(x) \leq \frac{x}{\log x}(\log\log x)^{O(1)}$. Improving on this, Saias [13] showed that there exist absolute constants c_1, c_2 such that

$$c_1\frac{x}{\log x} \leq P(x) \leq c_2\frac{x}{\log x}.$$

The most recent progress in this direction was made by Weingartner [23], who showed that there exists a positive constant c such that $P(x) \sim cx/\log(x)$ as $x \to \infty$.

An analog of the practical numbers arises in relation to divisors of polynomials of the form x^n-1. Recall that $x^n-1 = \prod_{d|n}\Phi_d(x)$, where $\Phi_d(x)$ is the dth cyclotomic polynomial, with $\deg\Phi_d(x) = \varphi(d)$. Notice that the degree of the right side is $\sum_{d|n}\varphi(d)$, which is equal to n. Thus, x^n-1 has a divisor of every degree less than or equal to n if and only if every number between 1 and n can be written as a sum $\sum_i \varphi(d_i)$ for distinct divisors $d_i \mid n$. Such integers n are now known as *φ-practical*.

Let $P_\varphi(x)$ denote the number of φ-practical numbers less than or equal to x. There are no Stewart-like criteria for determining whether a number n is φ-practical; however, in [22], the second author showed that there exist positive constants A, B such that

$$\frac{Ax}{\log x} \leq P_\varphi(x) \leq \frac{Bx}{\log x}.$$

In a subsequent paper with Pomerance and Weingartner, the second author [11] showed that there exists a positive constant C such that $P_\varphi(x) \sim Cx/\log x$ as $x \to \infty$.

Motivated by the studies of practical and φ-practical numbers, Schwab and the second author [16] generalized this construction to f-practical numbers for positive-integer-valued arithmetic functions f: a number n is f-practical if every integer between 1 and $S_f(n) = \sum_{d|n} f(d)$ can be written as a sum of $f(d)$'s, for distinct divisors d of n. Notice that $S_f(n)$ is the largest number that could be written as a sum of $f(d)$ where the values of d are distinct, so it is the natural upper bound for the interval where we can expect this property to hold. The original practical numbers and the φ-practical numbers correspond to the f-practical numbers for $f = \mathrm{id}$ and $f = \varphi$, respectively.

1.3. **Main results.** In this paper we will prove several results about the function

$$S_s(n) := \sum_{d|n} s(n).$$

In the spirit of the classical work of Davenport [2] on $n/\sigma(n)$ and Schoenberg [14, 15] on $\varphi(n)/n$, it is natural to consider whether the function $S_s(n)/n$ possesses a distribution function. We prove the following result in §4.

Theorem 4.4. The function $S_s(n)/n$ has a continuous asymptotic distribution function.

Schoenberg [14] also proved that the function $\varphi(n)/n$ has image dense in the interval $[0, 1]$. Analogous to this result, we prove the following.

Theorem 3.1. The values of $S_s(n)/n$ are dense in the interval $[0, \infty)$.

We also establish mean value computations for $S_s(n)$ and $S_s(n)/n$, and provide uniform bounds for the higher order moments of $S_s(n)/n$. In particular, we prove:

Theorem 5.4. The moments μ_k exist and are finite. Moreover, they satisfy

$$\log \mu_k \ll k \log\log k.$$

Our proofs mainly rely on standard tools from probabilistic number theory, which we outline in Section 2. However, the fact that $S_s(n)$ is neither additive nor multiplicative poses some additional challenges that we have found workarounds for. Moreover, it is not possible to use the classical analytic approach to prove that $S_s(n)/n$ has a continuous distribution function, due to the fact that the distribution function of $\log \sigma(n)/n$ is purely singular. Instead, we appeal to modern results of Lebowitz-Lockard and Pollack [6], which allow us to get around this problem.

2. Tools from probabilistic number theory

In this section, we introduce the definitions and tools from probabilistic number theory that will be used in our proofs in Sections 3, 4, and 5.

2.1. **Definitions and Notation.** A central concept in probabilistic number theory is that of asymptotic density, which is a formalization of the intuitive notion of the probability that an integer belongs to a set.

Definition 2.1. We define the *asymptotic density* (also called the *natural density* or simply *density*) of a subset $A \subset \mathbb{N}$ to be

$$\mathbf{d}A = \lim_{N\to\infty} \frac{\#\{a \in A : a \le N\}}{N},$$

when the limit exists. Replacing the limit by lim sup (resp. lim inf) yields the *upper density* (resp. *lower density*), which we denote $\overline{\mathbf{d}}$ (resp. $\underline{\mathbf{d}}$).

The asymptotic density can be seen as a limit of the probabilities $\mathbb{P}(n \in A)$ where n is restricted to the interval $[1, N]$. As such, asymptotic density preserves many nice properties of usual probabilities, but it does not form a measure on $\mathbb{N}$. In particular, the sets possessing an asymptotic density are not closed under countable union.

In classical probability theory, given a real random variable X following some distribution, the distribution function F associated to that distribution is $F(x) = \mathbb{P}(X \le x)$. Any function arising this way will be non-decreasing and right-continuous (i.e., $\lim_{x\to x_0^+} F(x) = F(x_0)$). Moreover, such a function will satisfy $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. We use these properties to define a general distribution function.

Definition 2.2. A non-decreasing function F is a distribution function (d.f.) if F is right-continuous and satisfies $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

For our purposes, a "random variable" will be an arithmetic function f. If the function f is well-behaved, then the function which appears will be a true distribution function according to the above definition.

Definition 2.3. Given an arithmetic function f, we define the sequence of functions

$$F_N(x) = \frac{\#\{n \leq N \colon f(n) \leq x\}}{N}.$$

We say f has asymptotic distribution function (a.d.f.) F if the functions F_N converge pointwise to a function F, and if F is a distribution function.

We note that if f has an a.d.f. F, then $F(x) = \mathbf{d}\{n : f(n) \leq x\}$.

Definition 2.4. For an arithmetic function f, we define the *mean value of f over $n \leq x$*, for x some positive real number, to be

$$M_x(f) = \frac{1}{x} \sum_{n \leq x} f(n).$$

Furthermore, we define the *mean value of f* to be $M(f) = \lim_{x \to \infty} M_x(f)$ when the limit exists.

Similarly, the *kth moment* of an arithmetic function f is defined to be

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)^k,$$

when the limit exists.

2.2. **Theorem of Erdős-Wintner.** Because of the utility of a.d.f.s, a rich theory has been established on the subject of when certain arithmetic functions have an a.d.f. A powerful theorem in this vein is the Erdős-Wintner Theorem [21, p.475], which completely answers the question of the existence of an a.d.f. in the case of additive arithmetic functions.

Theorem 2.5 (Erdős-Wintner, 1939)**.** Fix any real number $R > 0$. A real additive function $f(n)$ has a limiting distribution if and only if the following three series converge simultaneously:

$$\text{(i)} \sum_{|f(p)| > R} \frac{1}{p}; \qquad \text{(ii)} \sum_{|f(p)| \leq R} \frac{f(p)^2}{p}; \qquad \text{(iii)} \sum_{|f(p)| \leq R} \frac{f(p)}{p}.$$

In this case, all three sums converge for all $R > 0$. The limiting d.f. is either absolutely continuous, purely singular, or discrete. It is continuous if and only if

$$\sum_{f(p) \neq 0} \frac{1}{p} = \infty.$$

The Erdős-Wintner theorem gives insight not only into additive functions, but also multiplicative functions. If g is a strictly positive multiplicative function satisfying certain reasonable conditions[1], then g possesses an a.d.f. ψ if and only if the additive function $\log g$ possesses an a.d.f. ω. In this case, $\omega(x) = \psi(e^x)$.

Perhaps most surprising is that the Erdős-Wintner Theorem does not require considering $f(p^\alpha)$ for any $\alpha > 1$. In some sense, this tells us that if an additive function f has an a.d.f., then, for almost all n, the value of $f(n)$ is almost determined by its value on the squarefree part of n.

[1] g cannot be almost everywhere almost zero, i.e., it cannot be the case that for all $\varepsilon > 0$, $\mathbf{d}\{n \colon g(n) > \varepsilon\} = 0$. An example of a function failing this condition is $f(n) = 1/n$. See [1, Theorem 4].

As an application of the Erdős-Wintner theorem, one can prove the classical theorem of Davenport [2] that $n/\sigma(n)$ has a continuous distribution function. The same kind of argument can be applied to the functions $\varphi(n)/n$ and $n/S_\sigma(n)$ to show that they, too, have a.d.f.s.

Since $S_s(n)/n$ is not multiplicative, we cannot apply the Erdős-Wintner Theorem to yield an a.d.f. the way we can for the related functions $\sigma(n)/n$ and $S_\sigma(n)/n$. Moreover, for the function $f(n) = \log(S_s(n)/n)$, $f(p)$ is negative and unbounded, so there exists a prime p_0 so that $|f(p)| > R$ for all $p \geq p_0$. Thus, the sum (i) in the Erdős-Wintner theorem will diverge for this function. However, we will use the continuous distribution functions for $\sigma(n)/n$ and $S_\sigma(n)/n$ furnished by these theorems to show $S_s(n)/n$ has a continuous distribution function in Section 4.

3. $S_s(n)/n$ is dense in $\mathbb{R}^+$

In this section we will show that the values $S_s(n)/n$ are dense in $[0,\infty)$. First, we begin by recalling a classical result of Schoenberg [15]:

Theorem 3.1 (Schoenberg)**.** The values $n/\sigma(n)$ are dense in $[0,1]$.

Since the function $S_s(n)/n$ is not multiplicative, the argument Schoenberg used to prove Theorem 3.1 will not work. However, we are able to extract a version of Schoenberg's Theorem for $s(n)/n$ by writing it in terms of the function $\sigma(n)/n$. Namely, since $s(n)/n = \sigma(n)/n - 1$, it follows from Theorem 3.1 that the values of $s(n)/n$ are dense in $[0,\infty)$. One might hope that there is a similar representation for $S_s(n)/n$. For example, we can write

$$\begin{aligned} S_s(n) &= \sum_{d|n} s(d) \\ &= \sum_{d|n} (\sigma(d) - d) \\ &= \sum_{d|n} \sigma(d) - \sum_{d|n} d \\ &= S_\sigma(n) - \sigma(n). \end{aligned}$$

Then $S_s(n)/n = S_\sigma(n)/n - \sigma(n)/n$. However, determining whether the values of $S_s(n)/n$ are dense in $[0,\infty)$ from this seems to require that we be able to simultaneously control the growth of $S_\sigma(n)/n$ and $\sigma(n)/n$, which seems difficult.

To circumvent these problems, we introduce another relationship involving S_s: if a and b are relatively prime integers, then $S_s(ab) = S_s(a)S_s(b) + \sigma(a)S_s(b) + \sigma(b)S_s(a)$. To see this, we write

$$\begin{aligned} S_s(ab) &= S_\sigma(ab) - \sigma(ab) \\ &= S_\sigma(a)S_\sigma(b) - \sigma(a)\sigma(b) \\ &= (S_s(a) + \sigma(a))(S_s(b) + \sigma(b)) - \sigma(a)\sigma(b) \\ &= S_s(a)S_s(b) + \sigma(a)S_s(b) + \sigma(b)S_s(a). \end{aligned} \tag{1}$$

Observe that, when p is a prime, $S_s(p) = 1$. We will use this fact repeatedly throughout the remainder of this section. We now proceed with our result.

Theorem 3.2. The values $S_s(n)/n$ are dense in $[0,\infty)$.

Proof. Let $x \in [0,\infty)$, and index the primes in increasing order $p_1, p_2, \ldots$. If $x = 0$, then the sequence $S_s(p_i)/p_i = 1/p_i$ converges to x. Otherwise, $x > 0$. In this case, we break the result down into the following two claims.

Claim 1: if $N > 1$ is an integer such that $\frac{S_s(N)}{N} < x$ and so that every prime factor of N is smaller than

$$B(N) := \frac{S_\sigma(N)/N + S_s(N)/N}{x - S_s(N)/N},$$

then we can find a prime $q = q(N)$ with $B(N) < q < 2B(N)$, and so that

$$\frac{1}{2}\left(x + \frac{S_s(N)}{N}\right) < \frac{S_s(Nq)}{Nq} < x.$$

Claim 2: for any $x > 0$, there is an N satisfying the hypotheses of Claim 1.

To see how the theorem follows from the claims, fix x and let N be an integer satisfying the hypotheses of Claim 1. Starting from $N_1 = N$, we construct a sequence $N_1, N_2, \ldots$ by letting $N_{i+1} = N_i q(N_i)$. Then we have

$$0 < x - \frac{S_s(N_{i+1})}{N_{i+1}} < \frac{1}{2}\left(x - \frac{S_s(N_i)}{N_i}\right),$$

so the sequence $S_s(N_i)/N_i$ converges to x. (Notice that $B(N_{i+1}) \geq 2B(N_i) > q(N_i)$, so if N_i satisfies the hypotheses of Claim 1, then N_{i+1} does as well, and we are indeed able to build this sequence.)

To establish Claim 1, first notice that if N is an integer and q is a prime not dividing N, then by (1),

$$\begin{aligned}
\frac{S_s(Nq)}{Nq} &= \frac{S_s(N)S_s(q)}{Nq} + \frac{\sigma(N)S_s(q)}{Nq} + \frac{\sigma(q)S_s(N)}{Nq} \\
&= \frac{1}{q}\left(\frac{S_s(N)}{N} + \frac{\sigma(N)}{N}\right) + \frac{q+1}{q}\frac{S_s(N)}{N} \\
&= \frac{1}{q}\frac{S_\sigma(N)}{N} + \frac{q+1}{q}\frac{S_s(N)}{N} \\
&= \frac{1}{q}\left(\frac{S_\sigma(N)}{N} + \frac{S_s(N)}{N}\right) + \frac{S_s(N)}{N}. \qquad (2)
\end{aligned}$$

Now, given N satisfying the hypotheses of Claim 1, we must have that $B(N) \geq 2$ (since all the prime factors of N are less than $B(N)$), and so by Bertrand's Postulate we can find a prime q in the interval $(B(N), 2B(N))$. Such a q is coprime to N, so by the above computation

$$\frac{S_s(Nq)}{Nq} = \frac{1}{q}\left(\frac{S_\sigma(N)}{N} + \frac{S_s(N)}{N}\right) + \frac{S_s(N)}{N}.$$

Using that $q > B(N)$, we obtain

$$\begin{aligned}
\frac{1}{q}\left(\frac{S_\sigma(N)}{N} + \frac{S_s(N)}{N}\right) + \frac{S_s(N)}{N} &< \frac{1}{B(N)}\left(\frac{S_\sigma(N)}{N} + \frac{S_s(N)}{N}\right) + \frac{S_s(N)}{N} \\
&= \left(x - \frac{S_s(N)}{N}\right) + \frac{S_s(N)}{N} = x.
\end{aligned}$$

Further, using that $q < 2B(N)$, we obtain

$$\begin{aligned}\frac{1}{q}\left(\frac{S_\sigma(N)}{N}+\frac{S_s(N)}{N}\right)+\frac{S_s(N)}{N} &> \frac{1}{2B(N)}\left(\frac{S_\sigma(N)}{N}+\frac{S_s(N)}{N}\right)+\frac{S_s(N)}{N}\\ &= \frac{1}{2}\left(x-\frac{S_s(N)}{N}\right)+\frac{S_s(N)}{N} = \frac{1}{2}\left(x+\frac{S_s(N)}{N}\right),\end{aligned}$$

as desired.

Finally, we show that there exists an N satisfying the hypotheses of Claim 1. Fix $x > 0$, and let p_k be the least prime so that $S_s(p_k)/p_k < x$. (We can find such a prime since $S_s(p)/p = 1/p$.) For $m \geq k$, let $P_m = \prod_{i=k}^{m} p_i$. From (2), since $S_\sigma(N) \geq S_s(N)$, we have that $\frac{S_s(Nq)}{Nq} \geq \frac{q+2}{q}\frac{S_s(N)}{N}$ whenever q is a prime not dividing N. Applying this inequality repeatedly, we have that

$$S_s(P_m)/P_m \geq \frac{1}{p_k}\prod_{i=k+1}^{m}\frac{p_i+2}{p_i}.$$

Since the product $\prod_{i>k}\frac{p_i+2}{p_i}$ diverges, so too does $S_s(P_m)/P_m$, and so there exists an $m \geq k$ so that $S_s(P_m)/P_m < x \leq S_s(P_{m+1})/P_{m+1}$. Notice that

$$\begin{aligned}x-\frac{S_s(P_m)}{P_m} &\leq \frac{S_s(P_{m+1})}{P_{m+1}}-\frac{S_s(P_m)}{P_m}\\ &= \frac{S_s(p_{m+1}P_m)}{p_{m+1}P_m}-\frac{S_s(P_m)}{P_m}\\ &= \frac{1}{p_{m+1}}\left(\frac{S_\sigma(P_m)}{P_m}+\frac{S_s(P_m)}{P_m}\right).\end{aligned}$$

So, $B(P_m) \geq p_{m+1}$ while every prime factor of P_m is smaller than p_{m+1}, and so P_m satisfies the hypotheses of Claim 1. □

4. Continuous distribution function

We now prove that $S_s(n)/n$ has a continuous distribution function. Note that our approach differs from the classical analytic approach (c.f., [15], [14]) for an important reason. Using that $S_s(n)/n = S_\sigma(n)/n - \sigma(n)/n$, it is tempting to observe that $\log\sigma(n)/n$ and $\log S_\sigma(n)/n$ are additive functions with continuous distribution functions, and then apply the Erdős-Wintner Theorem to these distribution functions. However, it turns out that the distribution function for $\log\sigma(n)/n$ is purely singular, which makes it difficult to directly use these two distribution functions to create a distribution function for $S_s(n)/n$.

To get around this problem, we make use of modern technology that was recently introduced by Lebowitz-Lockard and Pollack [6]. If f is a real-valued arithmetic function, we say f *clusters around the real number* x if there exists a real number $d > 0$ such that for all $\varepsilon > 0$,

$$\overline{\mathbf{d}}\{n\colon x-\varepsilon < f(n) < x+\varepsilon\} \geq d.$$

If f does not cluster around any x, we say f is *nonclustering*. Suppose the arithmetic function f has an a.d.f. F. It is easy to see that if F is continuous then f is nonclustering. Recall that when F exists, it can be expressed as $\mathbf{d}\{n : f(n) \leq x\}$. Note that for any $\varepsilon > 0$ we have

$$\overline{\mathbf{d}}\{n\colon f(n) = x\} \leq \mathbf{d}\{n\colon x-\varepsilon < f(n) \leq x+\varepsilon\} = F(x+\varepsilon)-F(x-\varepsilon).$$

Since F is continuous, as $\varepsilon \to 0$ the right-hand side goes to 0. Thus, $\mathbf{d}\{n\colon f(n) = x\} = 0$. Indeed, the converse also holds.

Lemma 4.1. If the arithmetic function f has an a.d.f. F, and if f is nonclustering, then F is continuous.

Proof. Recall that F is the pointwise limit of the "partial" distribution functions F_N defined as

$$F_N(x) = \frac{\#\{n \le N : f(n) \le x\}}{N}.$$

Then, we have

$$\begin{aligned} F(x+\varepsilon) - F(x-\varepsilon) &= \lim_{N\to\infty} F_N(x+\varepsilon) - F_N(x-\varepsilon) \\ &= \mathbf{d}\{n : x-\varepsilon < f(n) \le x+\varepsilon\} \\ &\le \overline{\mathbf{d}}\{n : x-\varepsilon < f(n) < x+\varepsilon\}. \end{aligned}$$

Thus, by the assumption that f is nonclustering, as $\varepsilon \to 0$, we have $F(x+\varepsilon) - F(x-\varepsilon) \to 0$. Therefore, F is continuous. □

We will use the following two theorems, which appear as Theorem 1 and Proposition 5 in [6], respectively.

Theorem 4.2 (Lebowitz-Lockard and Pollack)**.** Let $f_1, ..., f_k$ be multiplicative arithmetic functions taking values in the nonzero real numbers and satisfying the following conditions:

(1) f_k does not cluster around 0
(2) for all $i < j$ with $i, j \in \{1, 2, ..., k\}$, the function f_i/f_j is nonclustering.
(3) for each i, whenever p and p' are distinct primes, we have $f_i(p) \ne f_i(p')$.

Then for all nonzero $c_1, ..., c_k \in \mathbb{R}$, the arithmetic function $F \coloneqq c_1 f_1 + \cdots c_k f_k$ is nonclustering.

Theorem 4.3 (Lebowitz-Lockard and Pollack)**.** Let $f_1, \ldots, f_k$ be positive-valued multiplicative functions each possessing a distribution function. Then for any $c_1, \ldots, c_k \in \mathbb{R}$, the function $c_1 f_1 + \cdots + c_k f_k$ also has a distribution function.

Both of these theorems are proven by explicit estimation of upper densities by using the arithmetic properties of the functions f_i. We now proceed with the proof of Theorem 4.4.

Theorem 4.4. The function $S_s(n)/n$ has a continuous a.d.f.

Proof. Recall that we can write

$$S_s(n) = \sum_{d|n} (\sigma(d) - d) = S_\sigma(n) - \sigma(n).$$

Thus,

$$\frac{S_s(n)}{n} = \frac{S_\sigma(n)}{n} - \frac{\sigma(n)}{n}$$

is a difference of two multiplicative functions.

Let $f_1 = S_\sigma(n)/n$, $f_2 = \sigma(n)/n$, and $F = f_1 + (-1)f_2$. We have previously stated that f_1 and f_2 have distribution functions, so by Theorem 4.3 above, F has an a.d.f. To show that the distribution function for F is continuous, by Lemma 4.1 it suffices to show that it satisfies the hypotheses of Theorem 4.2. We may apply Theorem 2.5 to the additive functions $\log f_1$, $\log f_2$, and $\log(f_1/f_2)$ to show that f_1, f_2 and f_1/f_2 have continuous a.d.f.s. Thus, conditions (1)-(3) of Theorem 4.2 are satisfied. Therefore, F is non-clustering. Since a distribution function for an arithmetic function F is continuous precisely when F is non-clustering, it follows that F is continuous. □

5. Mean values and moments of $S_s(n)/n$

In this section we will compute exact values and estimates of some common statistics for the function $S_s(n)/n$.

In the first subsection we will compute the mean values $M_x(S_s(n))$ and $M(S_s(n)/n)$. These results will ground our discussion in the following subsection of uniform estimates for the moments of $S_s(n)/n$.

5.1. **Mean values of $S_s(n)$ and $S_s(n)/n$.** To begin with, recall that by an elementary summation argument, $M_x(\sigma(n)) = \zeta(2)x/2 + O(\log x)$. We can use this fact to derive $M_x(S_\sigma(n))$ as follows:

$$\begin{aligned}
\frac{1}{x}\sum_{n\le x} S_\sigma(n) &= \frac{1}{x}\sum_{n\le x}\sum_{d|n}\sigma(d)\\
&= \frac{1}{x}\sum_{\substack{d,q\\ dq\le x}}\sigma(d)\\
&= \frac{1}{x}\sum_{q\le x}\sum_{d\le x/q}\sigma(d)\\
&= \frac{1}{x}\sum_{q\le x}\left(\frac{\zeta(2)}{2}\left(\frac{x}{q}\right)^2 + O\left(\frac{x\log x}{q}\right)\right).
\end{aligned}$$

From here it is a straightforward computation to verify that $M_x(S_\sigma(n)) = \zeta(2)^2x/2 + O((\log x)^2)$. We use these two values to compute the following result.

Theorem 5.1. The mean value $M_x(S_s(n))$ is given by

$$M_x(S_s(n)) = \frac{\zeta(2)(\zeta(2)-1)}{2}x + O((\log x)^2).$$

Proof. By linearity of M_x, we compute

$$\begin{aligned}
M_x(S_s(n)) &= M_x(S_\sigma(n) - \sigma(n))\\
&= M_x(S_\sigma(n)) - M_x(\sigma(n))\\
&= \frac{\zeta(2)(\zeta(2)-1)}{2}x + O((\log x)^2).
\end{aligned}$$

□

The following is an immediate corollary.

Corollary 5.2. We have

$$M(S_s(n)/n) = \zeta(2)(\zeta(2)-1).$$

Proof. Consider the sum $\sum_{n\le x} S_s(n)/n$. Applying partial summation with $a_n = S_s(n)$ and $f(n) = 1/n$ we find

$$\begin{aligned}
\sum_{n\le x}\frac{S_s(n)}{n} &= \frac{1}{x}\sum_{n\le x} S_s(n) + \int_1^x \frac{\sum_{n\le t} S_s(n)}{t^2}\,dt \\
&= M_x(S_s(n)) + \int_1^x \frac{M_t(S_s(n))}{t}\,dt \\
&= \frac{\zeta(2)(\zeta(2)-1)}{2}x + O((\log x)^2) + \int_1^x \left(\frac{\zeta(2)(\zeta(2)-1)}{2} + O((\log t)^2/t)\right)dt \\
&= \frac{\zeta(2)(\zeta(2)-1)}{2}x + O((\log x)^2) + \left(\frac{\zeta(2)(\zeta(2)-1)}{2}t\right)\Big|_1^x + O((\log t)^3\big|_1^x) \\
&= \zeta(2)(\zeta(2)-1)x + O((\log x)^3).
\end{aligned}$$

Thus, $M(S_s(n)/n) = \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} S_s(n)/n = \zeta(2)(\zeta(2)-1)$. □

5.2. **Estimates of the moments of $S_s(n)/n$.** In this section, we aim to estimate the moments of $S_s(n)/n$, i.e., the quantities

$$\mu_k = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(S_s(i)/i)^k.$$

We will make use of a powerful tool known as Wintner's Mean Value Theorem for multiplicative functions [12, Theorem 1, p. 138].

Theorem 5.3 (Wintner's Mean Value Theorem)**.** *If g is a multiplicative function satisfying*

(i) $$\sum_p \frac{|g(p)-1|}{p} < \infty$$

(ii) $$\sum_p \sum_{\nu=2}^{\infty} \frac{\left|g(p^\nu) - g(p^{\nu-1})\right|}{p^\nu} < \infty$$

then the mean value of g exists and is finite.

There are a few other facts we will make use of to establish our estimates for μ_k. We will use the following expressions for the functions σ and S_σ:

$$\sigma(p^\nu) = p^\nu\left(1+\frac{1}{p-1}\right) - \frac{1}{p-1}, \tag{3}$$

$$S_\sigma(p^\nu) = p^\nu\left(1+\frac{1}{p-1}\right)^2 - \frac{\nu+1}{p-1} - \frac{p}{(p-1)^2}. \tag{4}$$

We obtain these expressions by writing

$$\begin{aligned}
\sigma(p^\nu) &= \sum_{i=0}^{\nu} p^i \\
&= \frac{p^{\nu+1}-1}{p-1}.
\end{aligned}$$

Pulling out p^ν yields (1), and (2) follows from a similar argument.

Additionally, let μ'_k be the kth moment of the function $n/\varphi(n)$. We will use the estimates for μ'_k appearing in the proof of [7, Proposition 4.3], in particular,

$$\log \mu'_k \ll k \log\log k.$$

We may now proceed with the result.

Theorem 5.4. The moments μ_k exist and are finite. Moreover, they satisfy

$$\log \mu_k \ll k \log\log k.$$

Proof. First, the Binomial Theorem yields

$$\begin{aligned}(S_s(i)/i)^k &= \frac{(S_\sigma(i) - \sigma(i))^k}{i^k} \\ &= \frac{1}{i^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\sigma(i))^j(S_\sigma(i)^{k-j}).\end{aligned}$$

Each of the functions $h_{k,j}(i) = (\sigma(i))^j(S_\sigma(i))^{k-j}/i^k$ is multiplicative, and below we will use Wintner's Mean Value Theorem to show that each has finite mean. From the existence of mean values for the $h_{k,j}$, we conclude that the moments μ_k exist and are finite.

We first turn our attention to sum (i) in Theorem 5.3. Since $n \le \sigma(n) \le S_\sigma(n)$ for all n, we have that $0 \le h_{k,j}(p) - 1 \le h_{k,0}(p) - 1$, and so it suffices to check that sum (i) converges for $g = h_{k,0}$. Using expression (2), we get

$$\begin{aligned}h_{k,0}(p) - 1 &\le \left(\frac{S_\sigma(p)}{p}\right)^k - 1 \\ &< \left(1 + \frac{1}{p-1}\right)^{2k} - 1 \\ &= \frac{p^{2k} - (p-1)^{2k}}{(p-1)^{2k}} \\ &= \frac{p^{2k} - (p^{2k} - 2kp^{2k-1} + \text{terms of lower degree})}{(p-1)^{2k}} \\ &\ll_k \frac{p^{2k-1}}{(p-1)^{2k}} \\ &\ll_k \frac{1}{p}.\end{aligned}$$

Thus, for $g = h_{k,0}$, the summands in (i) are $O(1/p^2)$, so the sum converges.

For the double sum (ii), we fix k, j and use expressions (1) and (2) to estimate

$$\begin{aligned}h_{k,j}(p^\nu) &= \left(\frac{\sigma(p^\nu)}{p^\nu}\right)^j\left(\frac{S_\sigma(p^\nu)}{p^\nu}\right)^{k-j} \\ &= \left(\left(1 + \frac{1}{p-1}\right) + O\left(\frac{1}{p^{\nu+1}}\right)\right)^j\left(\left(1 + \frac{1}{p-1}\right)^2 + O\left(\frac{\nu}{p^{\nu+1}}\right)\right)^{k-j} \\ &= \left(1 + \frac{1}{p-1}\right)^{2k-j} + O\left(\frac{\nu}{p^{\nu+1}}\right).\end{aligned}$$

Thus, the numerator of the inner sum (ii) is $\left|h_{k,j}(p^\nu) - h_{k,j}(p^{\nu-1})\right| = O(\nu p^{-(\nu+1)})$. Therefore, the terms of the inner sum are $O(\nu p^{-(2\nu+1)})$. We can evaluate the series $S = \sum_{\nu=2}^{\infty} \frac{\nu}{p^{2\nu+1}}$ by using the geometric series $G = \sum_{\nu=2}^{\infty} x^{(2\nu+2)}$. We have $G = x^6/(1-x^2)$, so taking the derivative of both sides with respect to x yields

$$\begin{aligned}
\frac{6x^5 - 4x^7}{(1-x^2)^2} &= \frac{d}{dx}\sum_{\nu=2}^{\infty} x^{2\nu+2} \\
&= \sum_{\nu=2}^{\infty} (2\nu+2)x^{2\nu+1} \\
&= 2\left(\sum_{\nu=2}^{\infty} \nu x^{2\nu+1} + \sum_{\nu=2}^{\infty} x^{2\nu+1}\right).
\end{aligned}$$

Notice that the first term inside the parentheses becomes S when evaluated at $x = 1/p$, and the second term is geometric. Rearranging and solving for S gives us

$$S = \frac{2p^2 - 1}{(p^2-1)^2 p^3}.$$

So, we conclude that the inner sum converges to a value that is $O(p^{-5})$. Therefore, the double sum converges. Having checked that the hypotheses of Wintner's Mean Value Theorem hold, we conclude that each $h_{k,j}$ has a finite mean value.

By (2) above,

$$\begin{aligned}
S_\sigma(p^\nu)/p^\nu &= \left(1 + \frac{1}{p-1}\right)^2 - \frac{\nu+1}{p^\nu(p-1)} - \frac{1}{p^{\nu-1}(p-1)^2} \\
&\leq \left(1 + \frac{1}{p-1}\right)^2 \\
&= (p^\nu/\varphi(p^\nu))^2.
\end{aligned}$$

Since both $S_\sigma(n)/n$ and $(n/\varphi(n))^2$ are positive and multiplicative, we therefore have that $S_s(n)/n \leq S_\sigma(n)/n \leq (n/\varphi(n))^2$. So, we can use the estimates for $n/\varphi(n)$ to deduce that

$$\begin{aligned}
\log \mu_k &\leq \log \mu'_{2k} \\
&\ll 2k \log\log 2k \\
&\ll k \log\log k,
\end{aligned}$$

as desired. □

A consequence of Theorem 5.4 is yet another method of showing that $S_s(n)/n$ has a distribution function. By our computations above, we also have

$$\log \mu_{2k} \ll k \log\log k,$$

so there exists some index k_0 and constant A so that $\log \mu_{2k} \leq Ak \log\log k$ for all $k \geq k_0$. Hence, for all $k \geq k_0$ we have

$$\begin{aligned}
\mu_k &\leq \exp(Ak \log\log k) \\
&= (\log k)^{Ak}.
\end{aligned}$$

Therefore, for $k \geq k_0$,

$$\frac{\mu_{2k}^{1/2k}}{k} \leq \frac{(\log k)^{A/2}}{k}.$$

Thus, the condition $\limsup_{k\to\infty} \mu_{2k}^{1/2k}/k < \infty$ needed to apply Theorem 3.3.12 from [3] is satisfied, and therefore $S_s(n)/n$ has an a.d.f. As in Section 4, the results of Lebowitz-Lockard and Pollack suffice to show this a.d.f. is continuous.

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Department of Mathematics, University of Wisconsin, Madison, WI 53706
Email address: aidun@wisc.edu

Mathematical Institute, Universiteit Utrecht, 3584 CD Utrecht, The Netherlands
Email address: l.thompson@uu.nl