Preimages of the sum of proper divisor function

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The $s(n)$ function

Definition

Let $s(n)$ denote the sum of proper divisors of n.

Example: $s(p) = 1$ for any prime p

Example: $s(12) = 1 + 2 + 3 + 4 + 6 = 16$

We can write $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum-of-divisors function.

Perfect numbers

Pythagoras observed:

$$s(\mathbf{6}) = 1 + 2 + 3 = \mathbf{6}.$$

Definition

n is **perfect** if $s(n) = n$.

Augustine on perfect numbers

Augustine (400 CE): "Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because the number is perfect."

Amicable pairs

Definition

If $s(n) = m$, $s(m) = n$, and $m \neq n$, then n and m form an **amicable pair**.

Example (Pythagoras):

$$s(220) = 284, \quad s(284) = 220.$$

Amicable pairs

Definition

If $s(\mathsf{n}) = \mathsf{m}$, $s(\mathsf{m}) = \mathsf{n}$, and $\mathsf{m} \neq \mathsf{n}$, then n and m form an **amicable pair**.

Example (Pythagoras):

$$s(220) = 284, \quad s(284) = 220.$$

As of March 10, 2024, there are $1,228,889,024$ known amicable pairs!

Pythagoras on friendship

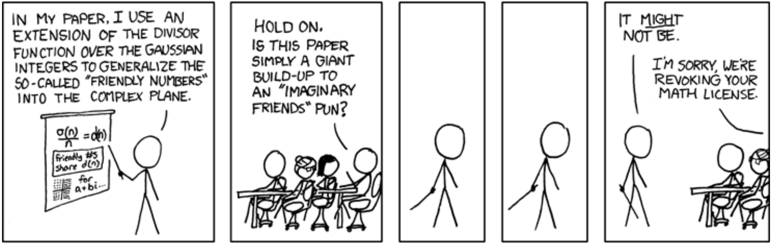
Pythagoras (6th century BCE), on friendship: "One who is the other I, such as 220 and 284."

Al-Majriti on amicable pairs

Al-Majriti (10th century CE): “[I] have tested the erotic effect of... giving any one the smaller number 220 to eat, and [myself] eating the larger number 284.”

A modern example

A modern example (from XKCD):



A friendship necklace

"Friendly numbers" necklace from MRCR Unique Creations

Iterates of s

We can view s as a dynamical system, looking at its iterates:

$6 \to 6$

$8 \to 7 \to 1$

$10 \to 8 \to 7 \to 1$

$12 \to 16 \to 15 \to 9 \to 4 \to 3 \to 1$

$28 \to 28$

$220 \to 284 \to 220$

$276 \to 396 \to \cdots$

A sequence of these iterates of s is known as an *aliquot sequence*.

Conjectures on the iterates of s

Catalan-Dickson Conjecture: Every aliquot sequence is bounded.

Guy-Selfridge Counter-Conjecture: Most aliquot sequences starting from an even number are unbounded.

No unbounded aliquot sequences are known, but the first candidate is 276.

Computational evidence

Evidence against Catalan-Dickson? Bosma looked at aliquot sequences with starting numbers below 10^6. Approximately $1/3$ of the even starters have yielded aliquot sequences that haven't yet terminated (computed up to 10^{99}).

Evidence against Guy-Selfridge? Bosma and Kane found that the asymptotic geometric mean of the ratios of $s(2n)/2n$ is slightly below 1.

Motivating questions

"Studying the comparison of $s(n)$ to n led to theorems of Schoenberg, Davenport, and Erdős-Wintner, and the birth of probabilistic number theory." -Carl Pomerance

In this talk, we will focus on two particular questions concerning the function $s(n)$:

1. Which numbers are of the form $s(n)$?
2. How large is the set $s^{-1}(n)$?

And then we will involve the integers with missing digits...

The image of s

Erdős was the first to consider questions about the image of s.

The image of s

It is easy to see that almost all odd numbers are contained in the image of s. To show this, we appeal to a variant of the Goldbach Conjecture that has been proven.

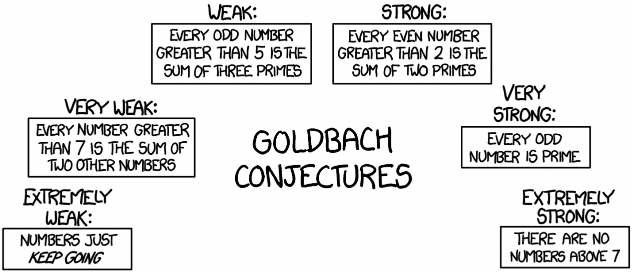


Photo credit: XKCD

Odd integers in the image of s

Theorem

Almost all odd numbers are contained in the image of s.

Proof.

$\square$

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If p, q are primes with $p \neq q$, then $s(pq) = p + q + 1$. □

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Strong Goldbach's Conjecture: All even integers ≥ 8 are the sum of two unequal primes.

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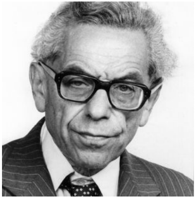
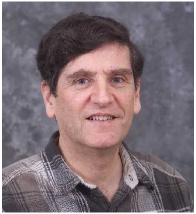
So almost all odd numbers ≥ 9 are values of s.

□

What about even numbers?





Theorem (Erdős, 1973)

A positive proportion of even integers are missing from the image of s.

Theorem (Luca & Pomerance, 2014)

A positive proportion of even integers are in the image of s.

The image of s

The function s can map sets of asymptotic density 0 to sets with positive asymptotic density.

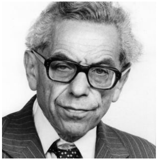
Example If $\mathcal{A} = \{pq : p, q \text{ prime}\}$ then $\mathcal{A}$ has asymptotic density 0 but $s(\mathcal{A})$ has asymptotic density $1/2$.

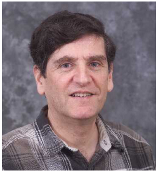
Example Erdős constructed sets $\mathcal{A}$ of positive density such that $s^{-1}(\mathcal{A})$ not only has density 0 but is, in fact, empty.

The preimage of s

What can be said about $s^{-1}(\mathcal{A})$ when $\mathcal{A}$ has asymptotic density 0?

The EGPS Conjecture





Conjecture (Erdős, Granville, Pomerance, Spiro, 1990)

Let $\mathcal{A}$ *be a set with asymptotic density* 0*. Then* $s^{-1}(\mathcal{A})$ *also has asymptotic density* 0*.*

Special cases of EGPS

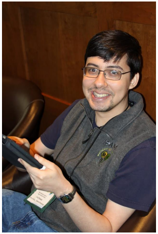
Some special cases of EGPS have been proven:

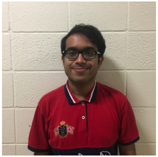
- (Pollack, 2014) If $\mathcal{A}$ is the set of primes, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.
- (Troupe, 2015)
  If $\mathcal{A}_\epsilon = \{m : |\omega(m) - \log\log m| > \epsilon \log\log m\}$ then $s^{-1}(\mathcal{A}_\epsilon)$ has asymptotic density 0.
- (Pollack, 2015) If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.
- (Troupe, 2020) If $\mathcal{A}$ is the set of integers that can be written as a sum of two squares, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.

Other recent related problems

Some very recent progress on $s(n)$:

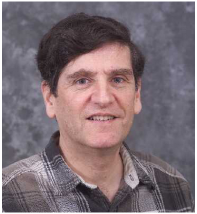
- (Pollack and Singha Roy, 2022) For any fixed $k \geq 4$, the k-th power-free values of n and $s(n)$ are equally common.
- (Lebowitz-Lockard, Pollack, Singha Roy, 2023) The values of $s(n)$ (for composite n) are equidistributed among the residue classes modulo p for small primes p.
- (Pollack and Troupe, 2023) The function $\omega(s(n))$ has the same mean and variance as $\omega(n)$.





Lebowitz-Lockard, Pollack, Singha Roy, and Troupe

Our result


Theorem (Pollack, Pomerance, T., 2017)

Let $\epsilon \to 0$ as $x \to \infty$. Suppose $\mathcal{A}$ is a set of at most $x^{1/2+\epsilon}$ positive integers. Then, as $x \to \infty$,

$$\#\{n \le x : s(n) \in \mathcal{A}\} = o_\epsilon(x)$$

uniformly in $\mathcal{A}$.

Consequences

Immediate consequences of our result:

- If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has density 0.
- If $\mathcal{A}$ is the set of squares, then $s^{-1}(\mathcal{A})$ has density 0.

Proof Sketch

We can assume that $\epsilon \geq 1/\log\log x$.

Let $\mathcal{A}$ be a set of at most $x^{1/2+\epsilon}$ integers.

When counting $m \leq x$ with $s(n) \in \mathcal{A}$, we can immediately discard inconvenient n, including:

- $n \leq x^{1/2}$
- n with no prime factor up to $\log x$
- n with squarefull part $> x^{2\epsilon}$
- n with $\gcd(n, \sigma(n)) > \log x$
- n with a divisor between $x^{1/2-10\epsilon}$ and $x^{1/2+10\epsilon}$

(With each of these conditions, we throw out $o(x)$ integers.)

Proof Sketch

Proof Strategy:

1. Show that for each $a \in \mathcal{A}$, the number of remaining $n \leq x$ with $s(n) = a$ is $\leq x^{1/2-\epsilon}$.

2. Since $\#\mathcal{A} \leq x^{1/2+\epsilon}$, this "pointwise" bound on the number of preimages is enough to complete the proof that

$$\#\{n \leq x : s(n) \in \mathcal{A}\} = o(x).$$

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Our assumptions on n imply that

$$d < x^{1/2-10\epsilon}$$

but also

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From these inequalities, one can deduce (using that n has small squarefull part) that $\gcd(d, e) = 1$.

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Moreover, this equation tells us that

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Given d, this puts $s(e)$ in a uniquely determined residue class modulo $s(d)/\gcd(s(d), \sigma(d))$.

Proof Sketch

Where are we at?

Given d, we want to count the number of possibilities for $s(e)$. We know that $s(e)$ is in a uniquely determine residue class mod $s(d)/\gcd(s(d), \sigma(d))$.

We want an upper bound on $s(e)$. A lower bound is easy: $s(e) \geq e/P^-(e)$.

The lower bound isn't so helpful, but it's not difficult to show that it isn't too far from the truth:

$$s(e) \ll \log x \cdot \frac{e}{P^-(e)}.$$

Proof Sketch

Recall:

$$s(e) \ll \log x \cdot \frac{e}{P^-(e)}.$$

Since $de = n \leq x$, we have $e \leq x/d$, so

$$s(e) \ll \log x \cdot \frac{x}{dP^-(e)}.$$

Remember $dP^-(e) \geq x^{1/2+10\epsilon}$, so

$$s(e) \ll \log x \cdot x^{1/2-10\epsilon}.$$

Proof Sketch

Recap: $s(e)$ is in a uniquely determined residue class modulo $s(d)/\gcd(s(d), \sigma(d))$ and $s(e) \ll \log x \cdot x^{1/2-10\epsilon}$.

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$$\ll (\log x)^3 \cdot x^{1/2-10\epsilon}/d + 1.$$

Summing over $d \leq x^{1/2-10\epsilon}$ gives our desired upper bound.

Integers with missing digits

Defining integers with restricted digits

For a proper subset $\mathcal{D} \subsetneq \{0, \dots, g-1\}$ such that $0 \in \mathcal{D}$, we define

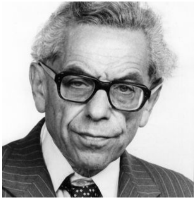
$$\mathcal{W}_{\mathcal{D}} := \left\{ n \in \mathbb{N} : n = \sum_{j \geq 0} \varepsilon_j(n) g^j, \varepsilon_j(n) \in \mathcal{D} \right\}$$

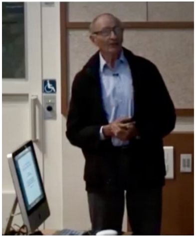
and

$$\mathcal{W}_{\mathcal{D}}(x) := \mathcal{W}_{\mathcal{D}} \cap [1, x] \, .$$

Notice that this set has asymptotic density 0.

Early results on integers with missing digits





Theorem (Erdős, Mauduit, and Sárközy, 1998)

Integers with missing digits are well-distributed in arithmetic progressions.

Almost primes with restricted digits

Theorem (Dartyge and Mauduit, 2000)

There exist infinitely many $n \in \mathcal{W}_{\{0,1\}}$ *with at most* $(1 + o(1))8g/\pi$ *prime factors as* $g \to \infty$.

Thin sets of primes

Maynard asked: Are there infinitely many primes with a given digit (e.g., 7) missing? Observe that:

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty$$

$$\sum_{p \text{ no } 7'\text{s}} \frac{1}{p} \le 100$$

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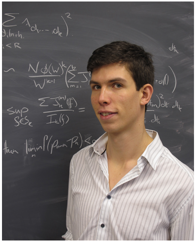
Extra challenge: applying sieve methods to "thin" sets.

Primes with missing digits



Theorem (Maynard, 2019)

There are infinitely many primes with missing digits.

Maynard's approach

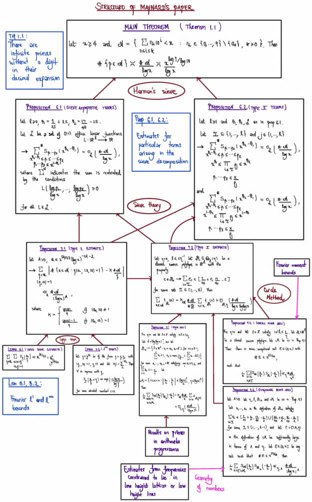
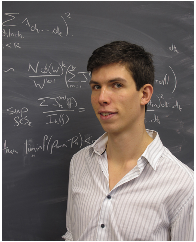


Diagram by Sebastían Carrillo Santana

Polynomial values with missing digits



Theorem (Maynard, 2022)

There are infinitely many n *such that* $P(n) \in \mathcal{W}_{\mathcal{D}}$*, for any given non-constant polynomial* $P \in \mathbb{Z}[X]$*, large enough base* g*, and* $\mathcal{D} = \{0, \ldots, g-1\} \setminus \{a_0\}$*.*

Sums of proper divisors with missing digits

Our WINE Project: we study $\mathcal{W}_{s,\mathcal{D}} := s^{-1}(\mathcal{W}_{\mathcal{D}})$.

Sums of proper divisors with missing digits

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2024)

Let $\mathcal{A}$ be a set of integers with missing digits in any base $g \geq 3$. Then $s^{-1}(\mathcal{A})$ has asymptotic density 0.

In other words, the EGPS Conjecture holds for sets of integers with missing digits!

An effective result

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2024)

Fix $g \geq 3$, $\gamma \in (0,1)$, and a nonempty subset $\mathcal{D} \subsetneq \{0, \ldots, g-1\}$. For all x sufficiently large, the number of $n \leq x$ for which $s(n)$ has all of its digits in base g restricted to digits in $\mathcal{D}$ is $O\left(\frac{x}{e^{(\log\log x)^{\gamma}}}\right)$.

How sharp is our bound?

Recall that $s(p) = 1$ for all primes p.

Then, if $\mathcal{D}$ contains 1, it follows that

$$\#\mathcal{W}_{s,\mathcal{D}}(x) \geq \pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

Thus, our result is optimal* for arbitrary $g, \mathcal{D}$.

*In the sense that γ cannot be replaced by a constant strictly greater than 1.

Sums of proper divisors with many missing digits

Recall:

Theorem (Pollack, Pomerance, T., 2017)

Let $\epsilon \to 0$ as $x \to \infty$. Suppose $\mathcal{A}$ is a set of at most $x^{1/2+\epsilon}$ positive integers. Then, as $x \to \infty$,

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If we remove at least half of the possible digits, then the size of this set of integers with missing digits is $O(\sqrt{x})$. Our 2017 result implies that the EGPS conjecture holds for this set.

An application of Maynard's work

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2024)

The function $s(n)$ takes infinitely many values in $\mathcal{W}_{\mathcal{D}}$.

Proof.

□

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Summary

In summary:

- Surprisingly little is known about $s(n)$ after millennia of study!
- EGPS conjectured that $s^{-1}(\mathcal{A})$ has asymptotic density 0 when $\mathcal{A}$ has asymptotic density 0.
- This has been confirmed for sets with specific structures (e.g., sets of integers with missing digits) and for sets of certain sizes ($O(x^{1/2+\varepsilon})$).
- The EGPS conjecture is still open.

Thank you!