SYSTOLES OF ARITHMETIC HYPERBOLIC SURFACES AND 3–MANIFOLDS

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ABSTRACT. Our main result is that for any positive real number x_0, the set of commensurability classes of arithmetic hyperbolic 2– or 3–manifolds with fixed invariant trace field k and systole bounded below by x_0 has density one within the set of all commensurability classes of arithmetic hyperbolic 2– or 3–manifolds with invariant trace field k. The proof relies upon bounds for the absolute logarithmic Weil height of algebraic integers due to Silverman, Brindza and Hajdu, as well as precise estimates for the number of rational quaternion algebras not admitting embeddings of any quadratic field having small discriminant. When the trace field is $\mathbf{Q}$, using work of Granville and Soundararajan, we establish a stronger result that allows our constant lower bound to instead grow with the area/volume. As an application, we establish a systolic bound for arithmetic hyperbolic surfaces that is related to prior work of Buser–Sarnak and Katz–Schaps–Vishne. Finally, we establish an analogous density result for commensurability classes of arithmetic hyperbolic 3–manifolds with a small area totally geodesic surface.

1. INTRODUCTION

Given a closed, orientable surface Σ_g of genus $g \geq 2$, the moduli space of hyperbolic metrics on Σ_g is denoted by $\mathscr{M}_g$. This $(3g-3)$ complex dimensional moduli space is a central object of interest in several fields. The present article is concerned with the subset of arithmetic hyperbolic points. It follows from work of Borel [1] that the set of arithmetic hyperbolic structures comprise a finite set in $\mathscr{M}_g$. These very special hyperbolic metrics naturally arise in connection to algebraic and geometric extremal problems; for instance, Hurwitz surfaces that achieve the maximal possible order isometry group are always arithmetic.

Associated to a hyperbolic metric is a discrete, faithful representation $\rho_{hol}\colon \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbf{R})$ with image that we denote by Γ. By seminal work of Margulis [19], a hyperbolic metric is arithmetic if and only if

$$[\mathrm{Comm}(\Gamma) : \Gamma] = \infty,$$

where

$$\mathrm{Comm}(\Gamma) = \{\eta \in \mathrm{PSL}(2,\mathbf{R}) \;:\; [\Gamma : \Gamma_\eta],\ [\Gamma^\eta : \Gamma_\eta] < \infty\}$$

and

$$\Gamma^\eta = \eta^{-1}\Gamma\eta, \quad \Gamma_\eta = \Gamma \cap \Gamma^\eta.$$

The group $\mathrm{Comm}(\Gamma)$ is referred to as the commensurator. There are several (conjectural) characterizations of arithmeticity based on algebraic information about Γ and geometric information about the metric itself (see Cooper–Long–Reid [6, 21], Geninska–Leuzinger [10], Lafont–McReynolds [15], Luo–Sarnak [17], and Schmutz [23]). These characterizations function through symmetries and it is not entirely clear what makes arithmetic hyperbolic surfaces special geometrically through the above lenses.

One well known conjecture regarding the special geometric nature of arithmetic hyperbolic surfaces is the Short Geodesic Conjecture. For a hyperbolic surface $M \in \mathscr{M}_g$, we denote the systole of M by $\mathrm{Sys}(M)$, and recall that this is the length of the shortest closed geodesic on M. The Short Geodesic Conjecture asserts that there exists a constant C, independent of genus, such that if M is an arithmetic hyperbolic surface, then $\mathrm{Sys}(M) \geq C$. There is an analogous conjectural uniform lower bound for the systole of arithmetic hyperbolic 3–manifolds. That Short Geodesic Conjecture for arithmetic hyperbolic 3–manifolds is slightly stronger than the Salem Conjecture (see [18, Section 12.3]) that asserts a uniform lower bound on the set of Mahler measures of the Salem polynomials.

In order to state our main results, we require some additional terminology. In Section 2 we will define the volume $V_{\mathscr{C}}$ of a commensurability class $\mathscr{C}$ of arithmetic manifolds. This volume is defined in terms of the volume of a distinguished representative of the class which arises in a natural way from a maximal order in the quaternion algebra associated to $\mathscr{C}$ and allows us to count the number of commensurability classes with bounded volume. Define $N_k(V)$ to be the number of commensurability classes of arithmetic hyperbolic 2–manifolds (respectively, 3–manifolds) with invariant trace field k and volume less than V. Given a positive real number x_0, define $N_k(V;x_0)$ to be the number of classes $\mathscr{C}$ with invariant trace field k, volume less than V, and which have a representative $M \in \mathscr{C}$ satisfying $\mathrm{Sys}(M) < x_0$.

Theorem 1.1. *For all sufficiently large x_0, we have*

$$N_k(V;x_0) \asymp V/\log(V)^{\frac{1}{2}}$$

as V tends to infinity, where the implied constants depend only on k and x_0.

By establishing that $N_k(V) \asymp V$, we deduce the following density result from Theorem 1.1.

Corollary 1.2. *For every $x_0 > 0$ and totally real number field k (respectively, number field with exactly one complex place), we have*

$$\lim_{V\to\infty} \frac{N_k(V;x_0)}{N_k(V)} = 0.$$

It is straightforward to see that there is a uniform lower bound for the systoles of arithmetic hyperbolic 2–manifolds with fixed invariant trace field k. As a result, for small values of x_0 we will have $N_k(V;x_0) = 0$ for all V, in which case the statement of Corollary 1.2 is trivially satisfied. However, for x_0 sufficiently large, $N_k(V;x_0)$ is unbounded. Our main result says that regardless of how large we fix our threshold for "short" with regard to systole, the density of commensurability classes that have a representative with a short geodesic is always zero.

Returning to $\mathscr{M}_g$, Theorem 1.1 says that the probability a commensurability class of k–arithmetic hyperbolic surfaces produces a point in the ε–thin part of M_g is asymptotically $1/\log(g)^{1/2}$ for any fixed ε. By ε–thin part, we mean the set of hyperbolic surfaces M with $\mathrm{Sys}(M) < \varepsilon$. By Mumford's Compactness Criterion, the associated thick-thin decomposition is analogous to the thick-thin decomposition of a hyperbolic n–manifold.

If we restrict to the class of arithmetic hyperbolic surfaces arising from quaternion algebras over $\mathbf{Q}$, we can allow our notion of short to grow with the area of the surface while still maintaining density one. Namely, with density one the systole has order of magnitude at least $\log\log(V_{\mathscr{C}})$.

Theorem 1.3. *Within the set of all commensurability classes of arithmetic hyperbolic surfaces with invariant trace field $\mathbf{Q}$ there is, for all $\varepsilon > 0$, a density one subset of classes $\mathscr{C}$ such that*

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma) > (\frac{1}{8} - \varepsilon)\log\log\left(\frac{24}{\pi}V_{\mathscr{C}}\right)$$

holds for all $\Gamma \in \mathscr{C}$.

If Γ is a maximal arithmetic lattice with invariant trace field $\mathbf{Q}$ which has minimal co-area in its commensurability class, one can deduce from Theorem 1.3 that with density one we have

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma) > (\frac{1}{8} - \varepsilon)\log\log\left(\frac{24}{\pi}V\right), \tag{1}$$

where V is the co-area of Γ. As the systole is non-decreasing in covers, we see for those commensurability classes, we have that as a uniform lower bound. In particular, we can compare (1) with the prior systolic estimates of Buser–Sarnak [3] and Katz–Schaps–Vishne [14]. Whereas previous systole bounds have made extensive use of careful trace estimates, we instead use counting arguments that take advantage of the basic tools of multiplicative number theory. One component of the argument is a bound for a negative moment of $L(1,\chi)$, as χ ranges over quadratic Dirichlet characters; we extract this bound from the detailed study made by Granville and Soundararajan [11] of the distribution of these L–values. Furthermore, our systole bounds do not have any unspecified additive constants. The presence of these constants in the work of Buser–Sarnak [3] and Katz–Schaps–Vishne [14] require that one pass to large congruence covers of an arithmetic hyperbolic surface in order to obtain a non-trivial systole bound. Our systole bounds on the other hand provide non-trivial bounds on the systole of maximal arithmetic lattices. In this light, our results should be seen as complementing those of Buser–Sarnak and Katz–Schaps–Vishne.

Remark. It is our use of the work of Granville–Soundararajan [11] that forces us to restrict to arithmetic surfaces with invariant trace field $\mathbf{Q}$.

One may view totally geodesic surfaces as an analogue of geodesics in a hyperbolic 3–manifold. With this motivation, our final result is an analogous density result for small area totally geodesic surfaces in commensurability classes of arithmetic hyperbolic 3–manifolds.

Theorem 1.4. *Let $V > 0$ and $k_1,\dots,k_r$ be the invariant trace fields of those arithmetic hyperbolic surfaces with area at most V. The set of commensurability classes of arithmetic hyperbolic 3–manifolds having a representative containing a totally geodesic surface with area at most V has density zero within the set of all commensurability classes of arithmetic hyperbolic 3–manifolds with invariant trace field a quadratic extension of some k_i.*

Obtaining uniform lower bounds on the area of the smallest totally geodesic surface is trivial since the area of *any* finite type hyperbolic surface is uniformly bounded from below. However, a surface can arise as a totally geodesic surface in infinitely many incommensurable manifolds and so the above density result is non-trivial.

The aforementioned density results are established using counting results that are of independent interest. In our prior work [16], the main input from analytic number theory came in the guise of Tauberian theorems for Dirichlet series. Such results give a convenient method for translating information about singular points into asymptotic estimates. The main novelty in this paper is the use of mean value estimates for multiplicative functions that are valid uniformly, instead of merely asymptotically. As with the counting results from [16], these methods potentially have a much broader range of applications to other algebraic and geometric counting problems, and subsequently broader geometric applications. Indeed, this paper serves as an illustration of these applications.

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2. Preliminaries

Notation. Throughout this paper k will denote a number field of signature $(r_1(k), r_2(k))$. In practice k will be either totally real or else contain a unique complex place. The ring of integers of k will be denoted by $\mathscr{O}_k$. Given an ideal I of $\mathscr{O}_k$, we will denote by $|I|$ its norm. The set of prime ideals of $\mathscr{O}_k$ will be denoted $\mathscr{P}_k$. The degree of k will be denoted by n_k, the discriminant by Δ_k, the associated Dedekind zeta function by $\zeta_k(s)$ and the regulator by Reg_k. If L/k is a finite extension then we will denote by $\Delta_{L/k}$ the relative discriminant.

Let k be a number field and B be a quaternion algebra over k. We will denote by $\mathrm{Ram}(B)$ the set of primes of k (possibly infinite) which ramify in B, by $\mathrm{Ram}_f(B)$ (respectively $\mathrm{Ram}_\infty(B)$) the subset of $\mathrm{Ram}(B)$ consisting of the finite (respectively infinite) primes of k which ramify in B. Similarly, we define the discriminant $\mathrm{disc}(B)$ of B to be the product of all primes (possibly infinite) in $\mathrm{Ram}(B)$. We define $\mathrm{disc}_f(B)$ and $\mathrm{disc}_\infty(B)$ similarly.

Let $\mathbf{H}^2$, $\mathbf{H}^3$ denote real hyperbolic space of dimension 2 and 3. We will denote by M an arithmetic hyperbolic 2– or 3–manifold and by Γ the associated arithmetic lattice in $\mathrm{PSL}(2,\mathbf{R})$ or $\mathrm{PSL}(2,\mathbf{C})$. In other words $M = \mathbf{H}^2/\Gamma$ or $M = \mathbf{H}^3/\Gamma$.

We will make use of standard asymptotic notation from analytic number theory throughout this paper. We will use interchangeably the Vinogradov symbol, $f \ll g$, and the Landau Big-Oh notation, $f = O(g)$, to indicate that there is a constant $C > 0$ such that $|f| \leq C|g|$. Moreover, we will write $f \asymp g$ to indicate that $f \ll g$ and $g \ll f$. Lastly, we write $f = o(g)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Any subscripts on these symbols will indicate dependence of the implied constants.

Arithmetic Manifolds. In this brief section we describe the construction of arithmetic lattices in $\mathrm{PSL}(2,\mathbf{R})$ and $\mathrm{PSL}(2,\mathbf{C})$. Our presentation will necessarily be terse. For a more detailed exposition we refer the reader to Maclachlan and Reid [18].

We begin by reviewing the construction of arithmetic Fuchsian groups. Let k be a totally real field and B a quaternion algebra over k which is unramified at a unique real place v of k. We therefore have an identification $B_v = B \otimes_k k_v \cong \mathrm{M}(2,\mathbf{R})$. Let $\mathscr{O}$ be a maximal order of B and $\mathscr{O}^1$ the multiplicative group consisting of those elements of $\mathscr{O}$ having reduced norm one. We denote by $\Gamma^1_{\mathscr{O}}$ the image in $\mathrm{PSL}(2,\mathbf{R})$ of $\mathscr{O}^1$. The group $\Gamma^1_{\mathscr{O}}$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$ having finite covolume. A subgroup Γ of $\mathrm{PSL}(2,\mathbf{R})$ is an **arithmetic Fuchsian group** if it is commensurable in the wide sense with a group of the form $\Gamma^1_{\mathscr{O}}$ for some totally real field k, quaternion algebra B over k and maximal order $\mathscr{O}$ of B. We will denote by $\mathscr{C}(k,B)$ the set of all discrete subgroups of $\mathrm{PSL}(2,\mathbf{R})$ commensurable with $\Gamma^1_{\mathscr{O}}$.

The construction of arithmetic Kleinian groups is very similar. Let k be a number field with a unique complex place v and B a quaternion algebra over k which is ramified at all real places. Let $\mathscr{O}$ be a maximal order of B and $\Gamma^1_{\mathscr{O}}$ the image in $\mathrm{PSL}(2,\mathbf{C})$ of $\mathscr{O}^1$ under the identification $B_v = B \otimes_k k_v \cong \mathrm{M}(2,\mathbf{C})$. A subgroup Γ of $\mathrm{PSL}(2,\mathbf{C})$ is an **arithmetic Kleinian group** if it is commensurable in the wide sense with a group of the form $\Gamma^1_{\mathscr{O}}$ for some number field k having a unique complex place, quaternion algebra B over k ramified at real primes and maximal order $\mathscr{O}$ of B.

Given two arithmetic lattices Γ_1, Γ_2 arising from (k_i, B_i), we know that Γ_1 and Γ_2 will be commensurable in the wide sense precisely when $k_1 \cong k_2$ and $B_1 \cong B_2$ [18, Theorem 8.4.1]. We will make use of this fact many times throughout the remainder of this paper.

Finally, throughout this paper we will be interested in counting the number of commensurability classes of arithmetic hyperbolic surfaces or 3–manifolds with a specified property. We will count these commensurability classes as follows. Let $\mathscr{C}(k,B)$ be a commensurability class of arithmetic hyperbolic surfaces or 3–manifolds. We define the **volume of** $\mathscr{C}(k,B)$ to be $V_{\mathscr{C}(k,B)} := \mathrm{covol}(\Gamma^1_{\mathscr{O}})$ where $\mathscr{O}$ is a maximal order in B. A result of Borel [1] (see also [18, Chapter 11.1]) shows that

$$V_{\mathscr{C}(k,B)} = \mathrm{covol}(\Gamma^1_{\mathscr{O}}) = \frac{8\pi\,|\Delta_k|^{\frac{3}{2}}\,\zeta_k(2)}{(4\pi^2)^{n_k}} \prod_{\mathfrak{p}|\mathrm{disc}_f(B)} (|\mathfrak{p}|-1)$$

when $\mathscr{C}(k,B)$ is a commensurability class of arithmetic hyperbolic surfaces and that

$$V_{\mathscr{C}(k,B)} = \mathrm{covol}(\Gamma^1_{\mathscr{O}}) = \frac{|\Delta_k|^{\frac{3}{2}}\,\zeta_k(2)}{(4\pi^2)^{n_k-1}} \prod_{\mathfrak{p}|\mathrm{disc}_f(B)} (|\mathfrak{p}|-1)\,.$$

when $\mathscr{C}(k,B)$ is a commensurability class of arithmetic hyperbolic 3–manifolds. Note that this definition does not depend on the choice of maximal order. It is with respect to this notion of volume that all of our counting results for commensurability classes of arithmetic manifolds will be based.

3. THEOREM 1.2: ABSOLUTE LOGARITHMIC WEIL HEIGHTS AND LENGTHS OF CLOSED GEODESICS

3.1. **Bounds for absolute logarithmic Weil heights.** In this section we count the number of commensurability classes of arithmetic hyperbolic 2– and 3–manifolds which have a fixed invariant trace field k and possess a representative with a closed geodesic of length less than x_0.

Our proof will make use of several important facts about absolute logarithmic Weil heights of algebraic integers, hence we begin by defining the relevant terms.

Let L be a number field, $\mathfrak{p} \in \mathscr{P}_L$ a prime ideal and $|\cdot|_{\mathfrak{p}}$ the associated valuation normalized so that for each $\alpha \in L$, we have

$$\prod_{\mathfrak{p}|\infty} |\alpha|_{\mathfrak{p}} = \left|\mathrm{Norm}_{L/\mathbf{Q}}(\alpha)\right|$$

and

$$\prod_{\mathfrak{p}\in\mathscr{P}_L} |\alpha|_{\mathfrak{p}} = 1.$$

We define the **logarithmic height of α relative to** L to be

$$h_L(\alpha) = \sum_{\mathfrak{p}\in\mathscr{P}_L} \log\left(\max\left\{1, |\alpha|_{\mathfrak{p}}\right\}\right).$$

The **absolute logarithmic Weil height of α** is

$$h(\alpha) = [L:\mathbf{Q}]^{-1} h_L(\alpha)$$

and is independent of the field L. We will make repeated use of the fact that **height of α relative to** $\mathbf{Q}(\alpha)$ is the logarithm of the Mahler measure of the minimal polynomial of α. The following height bounds will play an important role in the proof of this section's main result.

Theorem 3.1 (Silverman [26]). *Let L/k be a quadratic extension of number fields with norm of relative discriminant $\left|\Delta_{L/k}\right|$ and α be a primitive element for the extension. Then the absolute logarithmic Weil height $h(\alpha)$ of α satisfies*

$$h(\alpha) \geq \frac{-(r_1(k)+r_2(k))\log(2)}{2n_k} + \frac{1}{4n_k}\log\left|\Delta_{L/k}\right|.$$

Theorem 3.2 (Brindza [2], Hajdu [12]). *Let L be a number field of degree $n_L \geq 2$ with unit group rank r_L, absolute value of discriminant $|\Delta_L|$ and regulator Reg_L. Suppose further that L is not an imaginary quadratic field. Then there exists a system of fundamental units $\{u_1,\ldots,u_{r_L}\}$ of L such that the following inequality holds for all $1 \leq i \leq r_L$:*

$$h(u_i) \leq 6^{n_L} n_L^{5n_L} \,\mathrm{Reg}_L.$$

In order to proceed we now translate the height bounds of Theorems 3.1 and 3.2 into facts about the lengths of closed geodesics on certain arithmetic manifolds.

Proposition 3.1. *Let k be a totally real number field (respectively number field with a unique complex place) of degree n_k and B be a quaternion algebra over k which is unramified at precisely one real place of k (respectively ramified at all real places of k). If the commensurability class defined by the arithmetic data (k,B) possesses a representative with a primitive closed geodesic of length less than x_0 then there exists a quadratic extension L/k which embeds into B and satisfies $\left|\Delta_{L/k}\right| < e^{2(n_k+x_0)}$.*

Proof. We give a proof in the case that k has a unique complex place. The proof in the totally real case is similar and will be left to the reader. Suppose therefore that Γ is an arithmetic Kleinian group in the commensurability class defined by (k,B) such that the hyperbolic 3–orbifold $\mathbf{H}^3/\Gamma$ contains a closed geodesic of length $\ell(\gamma) < x_0$, where $\gamma \in \Gamma$ is the associated hyperbolic element. It is known [18, Chapter 8] that in this case the subgroup $\Gamma^{(2)}$ of Γ generated by squares is derived from a quaternion algebra in the sense that $\Gamma^{(2)}$ is contained in a group of the form $\Gamma^1_{\mathcal{O}}$ for some maximal order $\mathcal{O}$ of B. Note that $\Gamma^{(2)}$ contains the element γ^2, hence the quotient orbifold contains a closed geodesic of length $2\ell(\gamma)$. As the length of the closed geodesic associated to γ^2 is equal to twice the logarithm of the Mahler measure of the minimal polynomial of γ^2 [18, Lemma 12.3.3], which is equal to the height of γ^2 relative to $\mathbf{Q}(\lambda_{\gamma^2})$, we see that $[\mathbf{Q}(\lambda_{\gamma^2}):\mathbf{Q}] \in \{n_k, 2n_k\}$ (see [5, Lemma 2.3]) implies that the absolute logarithmic Weil height $h(\gamma^2)$ of γ^2 satisfies $x_0 > 2n_k h(\gamma^2)$. As the field $L = k(\lambda_{\gamma^2})$ is a quadratic extension of k which embeds into B (see [18, Chapter 12]), the proposition follows from Theorem 3.1. □

Proposition 3.2. *Let k be a totally real number field (respectively number field with a unique complex place) of degree n_k and absolute value of discriminant $|\Delta_k|$ and B be a quaternion algebra over k which is unramified at precisely one real place of k (respectively ramified at all real places of k). If B admits an embedding of a quadratic extension L/k which satisfies*

$$\left|\Delta_{L/k}\right| < \left(\frac{x_0}{4\cdot 6^{2n_k}(2n_k)^{10n_k+1}\left|\Delta_k\right|^{4n_k}}\right)^{1/2n_k}$$

and which is not totally complex in the case that k is totally real, then the commensurability class defined by the arithmetic data (k,B) possesses a representative with a primitive closed geodesic of length less than x_0.

Proof. As in the proof of Proposition 3.1 we will prove Proposition 3.2 in the case that k has a unique complex place and leave the totally real case to the reader.

As every real place of k ramifies in B, the Albert–Brauer–Hasse–Noether theorem implies that L is totally complex. It now follows from Dirichlet's unit theorem that every system of fundamental units of L/k contains a fundamental unit $u_0 \in \mathcal{O}_L^*$ such that $u_0^n \notin \mathcal{O}_k^*$ for any $n \geq 1$. Let σ denote the nontrivial automorphism of $\mathrm{Gal}(L/k)$ and define $u = u_0/\sigma(u_0)$. It is clear that $\mathrm{Norm}_{L/k}(u) = 1$ and that $u^n \notin \mathcal{O}_k^*$ for any $n \geq 1$.

Theorem 3.2 above, with Lemmas 4.4, 4.5 of [16] and elementary properties of Weil heights show that we may assume that u satisfies

$$h(u) \leq 2\cdot 6^{2n_k}(2n_k)^{10n_k}\left|\Delta_L\right|^{2n_k} \leq 2\cdot 6^{2n_k}(2n_k)^{10n_k}\left|\Delta_k\right|^{4n_k}\left|\Delta_{L/k}\right|^{2n_k}.$$

Let $\mathcal{O}$ be a maximal order of B which contains the quadratic order $\mathcal{O}_k[u]$ and let γ denote the image in $\Gamma^1_{\mathcal{O}}$ of u. The proposition now follows from Lemma 12.3.3 of [18] and the fact that the logarithm of the Mahler measure of the minimal polynomial of γ is at most $2n_k h(u)$. □

3.2. **A mean value theorem and applications.** Let k be a number field. A complex-valued function f defined on the (nonzero) ideals of $\mathcal{O}_k$ is *multiplicative* if $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b})$ whenever $\mathfrak{a}$ and $\mathfrak{b}$ are coprime. When $k = \mathbf{Q}$, the following mean value theorem appears as Theorem 01 in [13, Chapter 0], and the proof is sketched in Exercise 01 there. For details, see [25, p. 58]. The argument for general number fields k can be carried out in precisely the same way, simply by replacing the sums over natural numbers with sums over integral ideals, and so we omit it.

Proposition 3.3. *Let k be a number field. Let f be a nonnegative-valued multiplicative function on ideals of $\mathcal{O}_k$. Suppose there are constants $A \geq 0$ and $B \geq 0$ for which the following hold: For all $y \geq 0$,*

$$\sum_{|\mathfrak{p}|\leq y} f(\mathfrak{p})\log|\mathfrak{p}| \leq Ay, \tag{2}$$

and

$$\sum_{\mathfrak{p}} \sum_{\nu \geq 2} \frac{f(\mathfrak{p}^\nu)}{|\mathfrak{p}^\nu|} \log|\mathfrak{p}^\nu| \leq B. \tag{3}$$

(Here the sums on $\mathfrak{p}$ are over prime ideals of $\mathcal{O}_k$.) For all $x > 1$,

$$\begin{aligned}\sum_{|\mathfrak{a}| \leq x} f(\mathfrak{a}) &\leq (A+B+1) \frac{x}{\log x} \sum_{|\mathfrak{a}| \leq x} \frac{f(\mathfrak{a})}{|\mathfrak{a}|} \\ &\leq (A+B+1) \frac{x}{\log x} \prod_{|\mathfrak{p}| \leq x} \left(1 + \frac{f(\mathfrak{p})}{|\mathfrak{p}|} + \frac{f(\mathfrak{p}^2)}{|\mathfrak{p}^2|} + \dots \right).\end{aligned}$$

Suppose that at prime power ideals, the values of f are bounded by a constant depending at most on k. Then (2) and (3) hold with constants A and B depending only on k. Indeed, the bound

$$\sum_{|\mathfrak{p}| \leq y} \log|\mathfrak{p}| \ll_k y$$

is a crude consequence of Landau's prime ideal theorem. Moreover, since $\log|\mathfrak{p}^\nu| \ll |\mathfrak{p}|^{\nu/3}$,

$$\sum_{\mathfrak{p}} \sum_{\nu \geq 2} \frac{\log|\mathfrak{p}^\nu|}{|\mathfrak{p}^\nu|} \ll \sum_{\mathfrak{p}} \sum_{\nu \geq 2} |\mathfrak{p}|^{-2\nu/3} \ll \sum_{\mathfrak{p}} |\mathfrak{p}|^{-4/3}.$$

The final sum on $\mathfrak{p}$ is bounded above by $\zeta_k(4/3)$. Thus, we can choose a constant B, depending on k, such that (3) holds.

The following sieve lemma is a simple consequence of Proposition 3.3.

Lemma 3.4. *Let k be a number field, and let $\mathscr{P}$ be a set of prime ideals of $\mathcal{O}_k$. Suppose g is a nonnegative and multiplicative function on the ideals of $\mathcal{O}_k$, and that the values of g on prime power ideals are $O_k(1)$. Then for $x \geq 2$,*

$$\sum_{\substack{|\mathfrak{a}| \leq x \\ \mathfrak{a} \text{ squarefree} \\ \gcd(\mathfrak{a}, \mathscr{P}) = 1}} g(\mathfrak{a}) \ll_k x \prod_{\substack{|\mathfrak{p}| \leq x \\ \mathfrak{p} \notin \mathscr{P}}} \left(1 + \frac{g(\mathfrak{p}) - 1}{|\mathfrak{p}|}\right) \prod_{\substack{|\mathfrak{p}| \leq x \\ \mathfrak{p} \in \mathscr{P}}} \left(1 - \frac{1}{|\mathfrak{p}|}\right).$$

Here "$\gcd(\mathfrak{a}, \mathscr{P}) = 1$" denotes the condition that $\mathfrak{a}$ not be divisible by any member of $\mathscr{P}$.

Remark. The case when g is identically 1 is particularly important. In that case, Lemma 3.4 shows that the number of squarefree $\mathfrak{a}$ with $|\mathfrak{a}| \leq x$ and $\gcd(\mathfrak{a}, \mathscr{P}) = 1$ is

$$\ll_k x \prod_{\substack{|\mathfrak{p}| \leq x \\ p \in \mathscr{P}}} \left(1 - \frac{1}{|\mathfrak{p}|}\right). \tag{4}$$

Proof of Lemma 3.4. Let $f(\mathfrak{a}) = g(\mathfrak{a})\mu(\mathfrak{a})^2 \mathbf{1}_{\gcd(\mathfrak{a}, \mathscr{P})=1}$, where $\mathbf{1}_{\gcd(\mathfrak{a}, \mathscr{P})=1}$ denotes the charactistic function detecting those ideals for which $\gcd(\mathfrak{a}, \mathscr{P}) = 1$. Then f is multiplicative and its values at prime power ideals are $O_k(1)$. Applying Proposition 3.3,

$$\sum_{\substack{|\mathfrak{a}| \leq x \\ \mathfrak{a} \text{ squarefree} \\ \gcd(\mathfrak{a}, \mathscr{P}) = 1}} g(\mathfrak{a}) = \sum_{|\mathfrak{a}| \leq x} f(\mathfrak{a}) \ll_k \frac{x}{\log x} \prod_{\substack{|\mathfrak{p}| \leq x \\ \mathfrak{p} \notin \mathscr{P}}} \left(1 + \frac{g(\mathfrak{p})}{|\mathfrak{p}|}\right).$$

We now use the estimate

$$\frac{1}{\log x} \ll_k \prod_{|\mathfrak{p}| \leq x} \left(1 - \frac{1}{|\mathfrak{p}|}\right),$$

which is a crude version of Mertens' theorem for number fields. (For a sharper, asymptotic version, see [22].) Inserting this above gives

$$\sum_{\substack{|\mathfrak{a}|\le x\\ \mathfrak{a}\text{ squarefree}\\ \gcd(\mathfrak{a},\mathscr{P})=1}} g(\mathfrak{a}) \ll_k x\left(\prod_{\substack{|\mathfrak{p}|\le x\\ \mathfrak{p}\notin\mathscr{P}}}\left(1-\frac{1}{|\mathfrak{p}|}\right)\left(1+\frac{g(\mathfrak{p})}{\mathfrak{p}}\right)\right)\left(\prod_{\substack{|\mathfrak{p}|\le x\\ \mathfrak{p}\in\mathscr{P}}}\left(1-\frac{1}{|\mathfrak{p}|}\right)\right)$$

$$\le x\prod_{\substack{|\mathfrak{p}|\le x\\ \mathfrak{p}\notin\mathscr{P}}}\left(1+\frac{g(\mathfrak{p})-1}{|\mathfrak{p}|}\right)\prod_{\substack{|\mathfrak{p}|\le x\\ \mathfrak{p}\in\mathscr{P}}}\left(1-\frac{1}{|\mathfrak{p}|}\right). \qquad \square$$

Define a multiplicative function Φ on the non-zero integral ideals of k as follows:

$$\Phi(\mathfrak{a}) = |\mathfrak{a}|\prod_{\mathfrak{p}|\mathfrak{a}}\left(1-\frac{1}{|\mathfrak{p}|}\right).$$

Theorem 3.3. *Let k be a number field, and let L/k be a quadratic extension. Let $x\ge 2$. The number of quaternion algebras B over k with $\Phi(\mathrm{disc}_f(B))\le x$ and which admit an embedding of L is*

$$\ll_{k,L} \frac{x}{(\log x)^{1/2}}.$$

Proof. Since there are only $O_k(1)$ possibilities for $\mathrm{disc}_\infty(B)$, it is enough to establish the stated upper bound for the number of possible values of $\mathrm{disc}_f(B)$. Let $\mathscr{P}$ denote the set of prime ideals of k that split in L.

In what follows, we use $\mathfrak{d}$ to denote a squarefree ideal of $\mathscr{O}_k$ not divisible by any member of $\mathscr{P}$. Since $\mathrm{disc}_f(B)$ is such an ideal, it suffices to show the stated bound for the number of $\mathfrak{d}$ with $\Phi(\mathfrak{d})\le x$.

We begin by estimating a second moment. Using Lemma 3.4, we have for any $y\ge 3$ that

$$\sum_{|\mathfrak{d}|\le y}\left(\frac{|\mathfrak{d}|}{\Phi(\mathfrak{d})}\right)^2 \ll_k y\prod_{\substack{|\mathfrak{p}|\le y\\ \mathfrak{p}\notin\mathscr{P}}}\left(1+\frac{(|\mathfrak{p}|/\Phi(\mathfrak{p}))^2-1}{|\mathfrak{p}|}\right)\prod_{\substack{|\mathfrak{p}|\le y\\ \mathfrak{p}\in\mathscr{P}}}\left(1-\frac{1}{|\mathfrak{p}|}\right). \tag{5}$$

Now

$$\frac{(|\mathfrak{p}|/\Phi(\mathfrak{p}))^2-1}{|\mathfrak{p}|} = \frac{(2|\mathfrak{p}|-1)}{(|\mathfrak{p}|-1)^2|\mathfrak{p}|} = O\left(\frac{1}{|\mathfrak{p}|^2}\right).$$

Noting that $\sum_{\mathfrak{p}}\frac{1}{|\mathfrak{p}|^2}<\infty$ and that $1+t\le\exp(t)$ for all real t, we deduce that the first right-hand product in (5) satisfies

$$\prod_{\substack{|\mathfrak{p}|\le y\\ \mathfrak{p}\notin\mathscr{P}}}\left(1+\frac{(|\mathfrak{p}|/\Phi(\mathfrak{p}))^2-1}{|\mathfrak{p}|}\right)\le\exp\left(\sum_{\mathfrak{p}}O\left(\frac{1}{|\mathfrak{p}|^2}\right)\right)\ll_k 1.$$

To handle the second product, we can use the Chebotarev density theorem, according to which the primes of k that split in L have density $\frac{1}{2}$. From a version of that theorem with a reasonable error term (e.g., the version of the theorem given in [24]), along with partial summation, we have

$$\sum_{\substack{|\mathfrak{p}|\le y\\ \mathfrak{p}\in\mathscr{P}}}\frac{1}{|\mathfrak{p}|} = \frac{1}{2}\log\log|y| + O_{L,k}(1).$$

Since

$$\log\left(1-\frac{1}{|\mathfrak{p}|}\right) = -\frac{1}{|\mathfrak{p}|}+O\left(\frac{1}{|\mathfrak{p}|^2}\right),$$

it follows that the second right-hand product above is $O_{L,k}((\log y)^{-1/2})$, and so collecting everything,

$$\sum_{|\mathfrak{d}|\le y}\left(\frac{|\mathfrak{d}|}{\Phi(\mathfrak{d})}\right)^2 \ll_{k,L} \frac{y}{(\log y)^{1/2}}.$$

We now return to counting $\mathfrak{d}$ with $\Phi(\mathfrak{d})\le x$. Taking $y=x$ in the last estimate and noting that the summands are all at least 1, we see there are only $O(x/(\log x)^{1/2})$ possible $\mathfrak{d}$ with $|\mathfrak{d}|\le x$. Now let $y=2^\ell x$, where ℓ is a nonnegative integer. Observe that if $|\mathfrak{d}|>y$ but $\Phi(\mathfrak{d})\le x$, then

$$(|\mathfrak{d}|/\Phi(\mathfrak{d}))^2>(y/x)^2=4^\ell.$$

Hence,

$$\begin{aligned}\#\{\mathfrak{d}:|\mathfrak{d}|\in(y,2y],\Phi(\mathfrak{d})\le x\} &\le \frac{1}{4^\ell}\sum_{|\mathfrak{d}|\le 2y}\left(\frac{|\mathfrak{d}|}{\Phi(\mathfrak{d})}\right)^2\\ &\ll \frac{1}{4^\ell}\frac{2^{\ell+1}x}{\log(2^{\ell+1}x)^{1/2}} \ll \frac{1}{2^\ell}\frac{x}{(\log x)^{1/2}}.\end{aligned}$$

Summing on ℓ, we find that the total number of $\mathfrak{d}$ with $|\mathfrak{d}|>x$ but $\Phi(\mathfrak{d})\le x$ is also $O(x/(\log x)^{1/2})$. □

3.3. **Proof of Theorem 1.1.** Let x_0 be a positive real number and k be a number field which is totally real (respectively has a unique complex place). Recall that $N_k(V;x_0)$ is the number of commensurability classes of arithmetic hyperbolic 2–manifolds (respectively 3–manifolds) with invariant trace field k, volume less than V and containing a primitive closed geodesic of length less than x_0. To prove Theorem 1.1, we must show that for all sufficiently large x_0 we have

$$N_k(V;x_0)\asymp V/\log(V)^{\frac{1}{2}},$$

where the implied constants depend only upon k and x_0.

Proof of Theorem 1.1. We give a proof in the case of arithmetic hyperbolic 2–manifolds and leave the 3–manifold case to the reader. Borel [1, Section 7.3] has shown that the covolume of $\Gamma^1_{\mathscr{O}}$ is

$$\operatorname{covol}(\Gamma^1_{\mathscr{O}})=\frac{8\pi\,|\Delta_k|^{\frac{3}{2}}\,\zeta_k(2)\Phi(\operatorname{disc}_f(B))}{(4\pi^2)^{n_k}}, \tag{6}$$

where

$$\Phi(\operatorname{disc}_f(B))=\prod_{\mathfrak{p}|\operatorname{disc}_f(B)}(|\mathfrak{p}|-1).$$

It therefore follows from Proposition 3.1 that $N_k(V;x_0)$ is at most the number of isomorphism classes of quaternion algebras B over k which satisfy

$$\Phi(\operatorname{disc}_f(B))\le c_kV$$

and admit an embedding of some quadratic extension L/k with norm of relative discriminant

$$\left|\Delta_{L/k}\right|<e^{2(n_k+x_0)}.$$

Here c_k is a positive constant depending only on k and which can easily be made explicit via (6). A theorem of Datskovsky and Wright [7] shows that as $x\to\infty$ the number of quadratic extensions L/k with $\left|\Delta_{L/k}\right|<x$ is

$$\sim\frac{\kappa_k}{2^{r_2}\zeta_k(2)}x,$$

where κ_k is the residue at $s=1$ of the Dedekind zeta function $\zeta_k(s)$ of k and r_2 is the number of complex places of k.

Theorem 3.3 shows that for a fixed quadratic extension L of k, the number of quaternion algebras B over k with discriminant satisfying $\Phi(\mathrm{disc}_f(B)) < x$ and which admit an embedding of L is less than

$$\frac{\delta_L x}{\log(x)^{\frac{1}{2}}}$$

for some constant δ_L depending on L and k. Let $L_1, \dots, L_r$ be the quadratic extensions of k satisfying

$$\left|\Delta_{L_i/k}\right| < e^{2(n_k+x_0)}$$

and define

$$\delta := \max_{i=1,\dots,r} \delta_{L_i}.$$

The discussion above shows that for sufficiently large x_0,

$$\begin{aligned} N_k(V;x_0) &\ll_k e^{2x_0}\delta V/\log(V)^{\frac{1}{2}} \\ &\ll_{k,x_0} V/\log(V)^{\frac{1}{2}}. \end{aligned}$$

We now prove that for sufficiently large x_0,

$$N_k(V;x_0) \gg_{k,x_0} V/\log(V)^{\frac{1}{2}}.$$

Let x_0 be large enough that there exists a quadratic extension L of k which has signature $(2, n_k - 1)$ and satisfies

$$\left|\Delta_{L/k}\right| < \left(\frac{x_0}{4\cdot 6^{2n_k}(2n_k)^{10n_k+1}\left|\Delta_k\right|^{4n_k}}\right)^{1/2n_k}.$$

A minor modification to the proof of Theorem 1.7 of [16] shows that the number of quaternion algebras B over k which are unramified at a unique real place of k, admit an embedding of L and which satisfy $\left|\mathrm{disc}_f(B)\right| < V$ is $\gg V/\log(V)^{\frac{1}{2}}$ as $V \to \infty$. Proposition 3.2 shows that each commensurability class $\mathscr{C}(k,B)$ has a representative possessing a closed geodesic of length at most x_0. Observing that $\left|\mathrm{disc}_f(B)\right| > \Phi(\mathrm{disc}_f(B))$ we see that each of these classes $\mathscr{C}(k,B)$ satisfies

$$\begin{aligned} V_{\mathscr{C}(k,B)} &= \mathrm{covol}(\Gamma^1_{\mathscr{O}}) \\ &= \frac{8\pi\left|\Delta_k\right|^{\frac{3}{2}}\zeta_k(2)\Phi(\mathrm{disc}_f(B))}{(4\pi^2)^{n_k}} \\ &< \frac{8\pi\left|\Delta_k\right|^{\frac{3}{2}}\zeta_k(2)\left|\mathrm{disc}_f(B)\right|}{(4\pi^2)^{n_k}} \\ &\leq c_k\left|\mathrm{disc}_f(B)\right| \\ &\leq c_k V, \end{aligned}$$

where c_k is a positive constant which depends only on k. The theorem follows. □

Remark. We note that in our applications of results from [16] that one must take into account that in [16] the discriminant of a quaternion algebra B over k is defined to be the formal product of all infinite primes of k ramifying in B with the **square** of the product of all finite primes of k ramifying in B, whereas the present paper defines $\mathrm{disc}(B)$ to be the product of all primes of k (finite or infinite) ramifying in B.

3.4. **Proof of Corollary 1.2.** We are now ready to prove Corollary 1.2 using Theorem 1.1.

Proof of Corollary 1.2. In light of Theorem 1.1 it suffices to show that as $V \to \infty$, the number $N_k(V)$ of commensurability classes $\mathscr{C}$ of arithmetic hyperbolic 2–manifolds (3–manifolds) having invariant trace field k and $V_{\mathscr{C}} < V$ satisfies

$$N_k(V) \gg V, \tag{7}$$

where the implied constant depends only on k. Indeed, we would have

$$\frac{N_k(V;x_0)}{N_k(V)} \ll \frac{1}{\log(V)^{1/2}},$$

and hence verify the density zero claim. As above we will give a proof in the case of arithmetic hyperbolic 2–manifolds and leave the case of arithmetic hyperbolic 3–manifolds to the reader.

Let k be a totally real field, B a quaternion algebra over k in which a unique real place splits and $\mathscr{O}$ a maximal order of B. If for some $V > 0$, we have

$$\left|\operatorname{disc}_f(B)\right| < \frac{V(4\pi^2)^{n_k}}{8\pi\,|\Delta_k|^{\frac{3}{2}}\,\zeta_k(2)},$$

then it follows from (6) that we must have

$$V_{\mathscr{C}} := \operatorname{covol}(\Gamma^1_{\mathscr{O}}) < V, \tag{8}$$

where $\mathscr{C} = \mathscr{C}(k,B)$. A slight modification of Theorem 1.5 of [16] shows that as $V \to \infty$, the number $N_{k,quat}(V)$ of quaternion algebras B over k which are unramified at a unique real place of k and satisfy $\left|\operatorname{disc}_f(B)\right| < cV$ satisfies

$$N_{k,quat}(V) \gg V, \tag{9}$$

where the implied constant depends only on k. From (8) and (9), we obtain (7) as needed. □

Straightforward modifications to the proofs of Theorem 1.1, Theorem 3.3, and Corollary 1.2 show the following.

Corollary 3.5. *For any totally real number field k (respectively, number field with exactly one complex place), we have*

$$N_k(V) \asymp V,$$

where the implied constant depends upon k.

4. SYSTOLIC GROWTH OF ARITHMETIC HYPERBOLIC SURFACES

The main geometric goal of this section is the proof of Theorem 1.3

4.1. **Counting quaternion algebras into which few quadratic fields embed.** We start with a counting result that we will use with Proposition 3.1 in our proof of Theorem 1.3.

Theorem 4.1. *Let $h(x)$ be any function which is $o(\log(x)^{\frac{1}{2}})$, $N_{\mathbf{Q},quat}(x)$ be the number of quaternion algebras B over $\mathbf{Q}$ with $\operatorname{disc}_f(B) < x$ and $N'_{\mathbf{Q},quat}(x;h)$ be the number of quaternion algebras B over $\mathbf{Q}$ with $\operatorname{disc}_f(B) < x$ and which do not admit an embedding of any quadratic field having absolute value of discriminant less than $h(x)$. Then as $x \to \infty$,*

$$N'_{\mathbf{Q},quat}(x;h) \sim N_{\mathbf{Q},quat}(x).$$

To prove Theorem 4.1, we require the following lemma.

Lemma 4.1. *Let K be a quadratic field. The number $N_{\mathbf{Q},quat}(x;K)$ of quaternion algebras B over $\mathbf{Q}$ with $\operatorname{disc}_f(B) < x$ which admit an embedding of K satisfies*

$$N_{\mathbf{Q},quat}(x;K) \ll \frac{x}{(\log x)^{1/2}} \left(\frac{|\Delta_K|}{\phi(|\Delta_K|)} \right)^{1/2} \prod_{p \le x} \left(1 - \frac{\left(\frac{\Delta_K}{p}\right)}{p} \right)^{1/2}.$$

Here the implied constant is absolute.

Proof. Since B is an algebra over $\mathbf{Q}$, the square-free integer $\operatorname{disc}_f(B)$ determines B. Moreover, if K embeds into B, then $\operatorname{disc}_f(B)$ is not divisible by any prime that splits in K. Since $\operatorname{disc}_f(B) < x$, equation (4) — with $k = \mathbf{Q}$ and $\mathscr{P}$ the set of rational primes that split in K — shows that the number of possibilities for B is

$$\ll x \prod_{\substack{p \le x \\ p \nmid \Delta_K}} \left(1 - \frac{\left(1 + \left(\frac{\Delta_K}{p}\right)\right)/2}{p} \right) \le x \prod_{p \le x} \left(1 - \frac{\left(1 + \left(\frac{\Delta_K}{p}\right)\right)/2}{p} \right) \prod_{p \mid \Delta_K} \left(1 - \frac{1}{2p} \right)^{-1}. \tag{10}$$

Using that $\log(1+t) = t + O(t^2)$ for $|t| \le \frac{1}{2}$,

$$\log \left(1 - \frac{\left(1 + \left(\frac{\Delta_K}{p}\right)\right)/2}{p} \right) = \log \left(\left(1 - \frac{1}{p} \right)^{1/2} \left(1 - \frac{\left(\frac{\Delta_K}{p}\right)}{p} \right)^{1/2} \right) + O\left(\frac{1}{p^2} \right),$$

and similarly

$$\log \left(\left(1 - \frac{1}{2p} \right)^{-1} \right) = \log \left(\left(1 - \frac{1}{p} \right)^{-1/2} \right) + O\left(\frac{1}{p^2} \right).$$

Now exponentiate. Keeping in mind that

$$\prod_{p \le x} \left(1 - \frac{1}{p} \right) \ll \frac{1}{\log x},$$

that

$$\prod_{p \mid \Delta_K} \left(1 - \frac{1}{p} \right)^{-1} = \frac{|\Delta_K|}{\phi(|\Delta_K|)},$$

and that

$$\sum_p \frac{1}{p^2} < \infty,$$

we see that the final expression in (10) is

$$\ll \frac{x}{(\log x)^{1/2}} \left(\frac{|\Delta_K|}{\phi(|\Delta_K|)} \right)^{1/2} \prod_{p \le x} \left(1 - \frac{\left(\frac{\Delta_K}{p}\right)}{p} \right)^{1/2},$$

where the implied constants are absolute. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. Let H be a parameter assumed to satisfy $H \le \log x$. Let K be a quadratic field with $|\Delta_K| \le H$, and let $\chi(\cdot) = \left(\frac{\Delta_K}{\cdot}\right)$ be the associated quadratic Dirichlet character. Let us estimate the Euler product factor appearing in the upper bound of Lemma 4.1. Since

$$\log \left(1 - \frac{\chi(p)}{p} \right) = -\frac{\chi(p)}{p} + O\left(\frac{1}{p^2} \right)$$

and

$$\sum_p \frac{1}{p^2} < \infty,$$

we see that

(11) $$L(1,\chi)\prod_{p\le x}\left(1-\frac{\chi(p)}{p}\right)=\prod_{p>x}\left(1-\frac{\chi(p)}{p}\right)\ll\exp\left(-\sum_{p>x}\frac{\chi(p)}{p}\right).$$

Since χ is a primitive character of conductor $|\Delta_K|$, and $|\Delta_K|\le H\le\log x$, the prime number theorem for progressions implies that

$$\sum_{p\le T}\chi(p)\ll T/(\log T)^2,$$

uniformly for $T\ge x$. (Here we use [8, eq. (8), p.123], along with the bound

$$\beta_1<1-c/q^{1/2}\log^2 q$$

coming from Dirichlet's class number formula.) Hence, by partial summation,

$$\sum_{p>x}\frac{\chi(p)}{p}=O(1).$$

With (11), the above yields

$$\prod_{p\le x}\left(1-\frac{\chi(p)}{p}\right)\ll L(1,\chi)^{-1}.$$

It now follows from Lemma 4.1 that the number $N_{\mathbf{Q},quat}(H;K)$ of quaternion algebras $B/\mathbf{Q}$ that admit an embedding of some quadratic field K with $|\Delta_K|\le H$ satisfies

$$N_{\mathbf{Q},quat}(H;K)\ll\frac{x}{(\log x)^{1/2}}\sum_{|\Delta_K|\le H}\left(\frac{|\Delta_K|}{\phi(|\Delta_K|)}\right)^{1/2}L(1,\left(\frac{\Delta_K}{\cdot}\right))^{-1/2}.$$

By Cauchy–Schwarz, the sum on Δ_K is

$$\ll\left(\sum_{|\Delta_K|\le H}\frac{|\Delta_K|}{\phi(|\Delta_K|)}\right)^{1/2}\left(\sum_{|\Delta_K|\le H}L(1,\left(\frac{\Delta_K}{\cdot}\right))^{-1}\right)^{1/2}.$$

The first sum is $O(H)$. In fact, it is well-known that the arithmetic function $n/\phi(n)$ has finite moments of every order (see, e.g., [20, Exercise 14, p. 42]). From [11, Theorem 2] (with $z=-1$), and the subsequent comment there about Siegel's theorem, the second sum on Δ_K is also $O(H)$. We conclude that

$$N_{\mathbf{Q},quat}(H;K)\ll\frac{xH}{(\log x)^{1/2}}.$$

Finally, let $H=h(x)$. Since $h(x)=o((\log x)^{1/2})$, our upper bound is $o(x)$. Since $N_{\mathbf{Q},quat}(x)\gg x$ from [16, Theorem 1.5], the theorem follows. □

Remark. The above argument is similar to the proof of [9, Lemma 2.6].

4.2. **Applications to the systole growth of arithmetic hyperbolic surfaces.** Recall that for a hyperbolic 2–orbifold M, we denote the systole of M by $\mathrm{Sys}(M)$, which is the length of the shortest closed geodesic on M.

Theorem 4.2. *Let S^1 be the set of all arithmetic Fuchsian groups of the form $\Gamma^1_{\mathcal{O}}$ where $\mathcal{O}$ is a maximal order of an indefinite quaternion division algebra over $\mathbf{Q}$ and S^{min} be the set of all maximal arithmetic Fuchsian groups with invariant trace field $\mathbf{Q}$ which have minimal covolume within their commensurability class. Then for all $\varepsilon>0$ the following are true:*

(i) The set of $\Gamma\in S^1$ such that

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma)>(\frac{1}{4}-\varepsilon)\log\log\left(\frac{3}{\pi}\mathrm{Vol}(\mathbf{H}^2/\Gamma)\right)$$

has density one in S^1.

(ii) The set of $\Gamma \in S^{min}$ *such that*

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma) > (\frac{1}{4} - \varepsilon) \log\log\left(\frac{24}{\pi}\,\mathrm{Vol}(\mathbf{H}^2/\Gamma)\right)$$

has density one in S^{min}.

Proof. Let $h(x) = \log(x)^{\frac{1}{2}-\varepsilon}$ and B be an indefinite quaternion division algebra over $\mathbf{Q}$ which does not admit an embedding of any quadratic field with absolute value of discriminant less than $h(\mathrm{disc}_f(B))$. Note that by Theorem 4.1 and its proof, the set of such quaternion algebras has density one within the set of all indefinite quaternion division algebras over $\mathbf{Q}$. Let $\mathscr{O}$ be a maximal order in B so that $\Gamma^1_{\mathscr{O}} \in \mathscr{C}(\mathbf{Q}, B)$ and Γ^{min} be the element of $\mathscr{C}(\mathbf{Q}, B)$ with minimal covolume. We remark that because every indefinite quaternion algebra defined over $\mathbf{Q}$ has type number 1, there is a one to one correspondence between arithmetic Fuchsian groups of the form $\Gamma^1_{\mathscr{O}}$, arithmetic Fuchsian groups of the form Γ^{min} and commensurability classes $\mathscr{C}(\mathbf{Q}, B)$. The first assertion now follows from Proposition 3.1, Theorem 4.1 and equation (13). The second assertion follows from the same reasoning along with the fact that (see [1] and Lemma 2.1 of [4])

$$[\Gamma^{min} : \Gamma^1_{\mathscr{O}}] = 2^{1+\#\mathrm{Ram}_f(B)}.$$

□

As a consequence of Theorem 4.2, we obtain Theorem 1.3 from the introduction. The details are as follows.

Proof of Theorem 1.3. In light of Theorem 4.2 it suffices to show that for all $\Gamma \in \mathscr{C}(\mathbf{Q}, B)$, where $\mathscr{C}(\mathbf{Q}, B)$ lies within a set of commensurability classes of density one,

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma) > \frac{1}{2}\,\mathrm{Sys}(\mathbf{H}^2/\Gamma^{min}).$$

To that end, let $\gamma \in \Gamma$ be a hyperbolic element whose associated geodesic has length $\ell(\gamma)$. The subgroup $\Gamma^{(2)}$ generated by squares of elements in Γ is contained in $\Gamma^1_{\mathscr{O}}$ for some maximal order $\mathscr{O}$ of B (see [18, Chapter 8]). As Γ^{min} is derived from the normalizer $N(\mathscr{O})$ of $\mathscr{O}$ in B^*, we see that up to isomorphism $\gamma^2 \in \Gamma^{(2)} \subset \Gamma^1_{\mathscr{O}} \subset \Gamma^{min}$. As $\ell(\gamma^2) = 2\ell(\gamma)$, the corollary follows from Theorem 4.2. □

We now provide more details on the relation of Theorem 1.3 with the work of Buser-Sarnak. In [3], they proved that if Γ is a cocompact arithmetic Fuchsian group defined over $\mathbf{Q}$ then there is a constant $c = c(\Gamma)$ such that systole of the congruence cover $\Gamma[I]$ satisfies

$$\mathrm{Sys}(\mathbf{H}^2/\Gamma[I]) > \frac{4}{3}\log\left(g(\mathbf{H}^2/\Gamma[I])\right) - c, \tag{12}$$

where $g(\cdot)$ denotes genus. This result was subsequently extended to arbitrary cocompact arithmetic Fuchisan groups by Katz, Schaps and Vishne [14]. Note that while the work of Katz, Schaps and Vishne also contains an unspecified constant, this constant was made explicit in certain special cases; for instance when Γ is the fundamental group of a Hurwitz surface. The methods used in both papers are similar and involve extensive use of trace estimates. Theorem 1.3 provides a density one lower bound with order of magnitude $\log\log$, which is clearly not as good as (12) which has order of magnitude log. On the other hand, our lower bound holds without a depth requirement, in contrast to the result above that becomes non-trivial only once the level is sufficiently big. Important here is that our method provides non-trivial bounds on the systole growth of a density one subset of commensurability classes of maximal arithmetic hyperbolic surfaces. That yields a depth free bound on a density one set of classes since systole is non-decreasing when passing to a finite cover.

5. Theorem 1.4: Totally geodesic surfaces of small area

Theorem 1.4 is an immediate consequence of the following theorem.

Theorem 5.1. *Let k be a totally real field and $\mathscr{C}(k,B_0)$ a commensurability class of arithmetic hyperbolic surfaces with invariant trace field k. The set of commensurability classes of arithmetic hyperbolic 3–manifolds with a representative containing a totally geodesic surface in $\mathscr{C}(k,B_0)$ has density zero within the set of all commensurability classes of arithmetic hyperbolic 3–manifolds with invariant trace field a quadratic extension of k.*

Proof. Recall that the volume of a commensurability class $\mathscr{C} = \mathscr{C}(L,B)$ of arithmetic hyperbolic 3–manifolds with invariant trace field L is

$$V_{\mathscr{C}} = \mathrm{covol}(\Gamma^1_{\mathscr{O}}) = \frac{|\Delta_L|^{\frac{3}{2}}\,\zeta_L(2)\Phi(\mathrm{disc}_f(B))}{(4\pi^2)^{n_k-1}}, \tag{13}$$

where $\mathscr{O}$ is a maximal order of B. The results of [18, Chapter 9.5] show that a representative of $\mathscr{C} = \mathscr{C}(L,B)$ contains a primitive totally geodesic surface in $\mathscr{C}(k,B_0)$ if and only if L is a quadratic field extension of k with a unique complex place and $B_0 \otimes_k L \cong B$. The theorem of Datskovsky and Wright [7] shows that the number of such $\mathscr{C}(L,B)$ with $V_{\mathscr{C}(L,B)} < V$ is $\ll V^{\frac{2}{3}}$ for large V, where the implied constant depends on $\mathscr{C}(k,B_0)$.

Suppose now that L is a fixed quadratic extension of k which has a unique complex place. Then there exists a constant $\delta_L > 0$ such that the number of quaternion algebras B over L such that $V_{\mathscr{C}(L,B)} < V$ is $\gg \delta_L V$ for large V (see [16, Theorem 1.5] and the remark which follows the proof of Theorem 1.1), which already proves the theorem. □

The following is an immediate consequence of Theorem 5.1, the fact that there are only finitely many arithmetic hyperbolic surfaces of bounded volume [1] and the fact that all totally geodesic surfaces of an arithmetic hyperbolic 3–manifold must have the same invariant trace field (which must be the maximal totally real subfield of the invariant trace field of the 3–manifold).

Corollary 5.1. *For all $V > 0$ the set of commensurability classes of arithmetic hyperbolic 3–manifolds with a representative containing a totally geodesic surface with area less than V has density zero within the set of all commensurability classes of arithmetic hyperbolic 3–manifolds.*

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