WISM559
SEMINAR NUMBER THEORY Spring 2022: Arithmetic Manifolds

Organized by Gunther Cornelissen & Lola Thompson

Important! *Interested participants should contact Gunther Cornelissen by email* (g.cornelissen@uu.nl), *outlining their interest in the seminar, specific topics they are interested in, and their familiarity with prerequisites*.

This seminar is about arithmetic manifolds. Such manifolds are constructed as quotients of an “easy” space by an action of an interesting “arithmetic” group. In the two-dimensional world, it involves, for example, the construction of modular curves. These are ubiquitous in number theory because of their connection with modular forms; for example, the geometry of modular surfaces can be used to compute dimensions of spaces of modular forms. In the three-dimensional case – the one that we will focus on - they relate to the arithmetic of quaternion algebras. Three-dimensional manifolds *M* come in many flavours (related, e.g., to knot theory) and are very mysterious in general. However, when their construction arises from number theory, we can use tools from algebraic and analytic number theory to better understand them (for example, the existence of immersed surfaces in *M* depends on ramification properties of an associated number field; the volume of *M* relates to special values of zeta-functions; etc.).

Contents The precise direction of the course is very much dependent on the participants, but here is a preliminary list of possible topics:

- Construction of modular curves and hyperbolic 3-manifolds as quotient spaces for group actions of modular groups, Fuchsian groups, Kleinean groups.
- Commensurability and arithmeticity. Relation to (Mostow) rigidity.
- Quaterion algebras and their arithmetic: orders, units, class number, type number.
- The trace field, a number field that provides invariants of an arithmetic manifold. Using the trace field in connection with embedded surfaces and geodesics.
- Examples of arithmetic manifolds: Bianchi groups, link complements, the Weeks manifold.
- Volumes of (arithmetic) hyperbolic manifolds and number theory; estimating volumes; Borel’s theorem, “smallest” manifolds. Related counting problems in analytic number theory.
- Shortest geodesic conjecture; relation to Lehmer’s problem in number theory.
- Construction of isospectral manifolds; the isospectrality problem; counting problems.
- Growth of homology rank and torsion for arithmetic manifolds; arithmetic conjectures.
- Computer programs for hyperbolic arithmetic 3-manifolds: Snap/SnapPy.

Material
The following two books contain basic material on arithmetic manifolds; we will use some chapters in the second book to introduce the basics of (hyperbolic) manifolds, and then treat selected material on quaternion algebras and number theoretical aspects using both sources:

- Reid and C. Maclachlan, *The Arithmetic of Hyperbolic 3-Manifolds*, Graduate Text in Math. Vol. 219, Springer-Verlag (2003)

- J. Voight, *Quaternion algebras*. Graduate Texts in Math. Vol. 288, Springer-Verlag (2021) [freely available online]

The following further references are illustrations of other typical results. This is just a small selection from a huge amount of literature.

- M. Belolipetsky, *Hyperbolic orbifolds of small volume*, Proceedings of the International Congress of Mathematicians-Seoul 2014, Vol. II, 837-851 (2014).
- N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu vol. 12, no. 2, 391-447 (2013).
- E. Breuillard and B. Deroin, *Salem numbers and the spectrum of hyperbolic surfaces*, Int. Math. Res. Not. IMRN, vol. 22, 8234-8250 (2020).
- T. Chinburg, E. Friedman, K.N. Jones and A. Reid, *The arithmetic hyperbolic 3-manifold of smallest volume*, Ann. Scuola Normale Superiore Pisa – Cl. di Sci., Série 4, vol. 30, no. 1, 1-40 (2001).
- G. Cornelissen and N. Peyerimhoff, *Twisted isospectrality, homological wideness and isometry*, preprint arXiv:2107.00253 (2021; 49pp.)
- B. Linowitz, D.B. McReynolds, P. Pollack, L. Thompson, *Counting and effective rigidity in algebra and geometry*, Invent. Math. 213, no. 2, 697-758 (2018)
- M.-F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques*, Ann. of Math. (2), no. 1, vol. 112, 21-32 (1980)
- C.D. Hodgson and J.R. Weeks, *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds,* Experiment. Math. Vol. 3, no. 4, 261-274 (1994)

Prerequisites Basic linear algebra, group theory, and rings are required, as well as familiarity with basic point set topology and algebraic number theory. Knowledge of the theory of (topological or differentiable) manifolds is useful for some parts, but not required, and aspects of this can be recalled during the seminar, depending on the participants.

Format This seminar is aimed at master's and graduate students with some background in algebra, number theory and basic topology. Other interested participants can volunteer to give a talk, depending on availability. There is a maximal number of (active) participants to the seminar of 12, with priority given to the aforementioned students.

ECTS: 7.5.

Schedule: Second Semester, blocks 3+4, 2 x 45 minutes per week

Language: English

Evaluation: Participants are expected to give two seminar talks (i.e., 2 x 45 minute presentations), possibly more or less depending on the number of participants. They will study the material beforehand, hold a blackboard presentation about it, and make a handout. They will pose a hand-in exercise to the other seminar participants (to be handed in at the next lecture) that should be approved beforehand by the seminar organizers. The speaker is responsible for grading this hand-in exercise and providing sufficient feedback to the other students. In case of a dispute over correctness of a solution or grading, the seminar organizers are the final arbitrators. It is obligatory to be present at all talks in this seminar (except in cases of *force majeure*). The final grade for the seminar is based on the average grade that the seminar organizers give to your talks, handouts, homework preparation and corrections (70%), and on your homework grades

(30%).

Learning goals After completing the course, the student will be able to:

- convert material from part of a graduate-level textbook or scientific paper into a coherent and comprehensible presentation for fellow students and mathematicians in general.
- choose appropriate means of communicating theoretical mathematics to fellow students and mathematicians, in written and oral form.
- Formulate and correct exercises that maintain a balance between relevance, interest, and feasibility.
- explain specific topics from the content list of the seminar to fellow students, and put them into context as far as their relevance to wider mathematics is concerned.

"Toetsmatrijs"
Presentation(s) = presentation and participation in lectures
Homework = combined output for homework assignments

	Presentation (s)	Homewo rk
Understanding the material	30	0
Effective communication of the material	30	0
Formulating and correcting homework	0	10
Homework grades	0	30