The sign changes of Fourier coefficients of Eisenstein series

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Structure of modular forms

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$$M_k(N, \chi) = S_k(N, \chi) \oplus E_k(N, \chi),$$

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Moral: Every modular form can be written uniquely as the sum of a cusp form and an Eisenstein series.

Hecke operators

Hecke operators are a family of operators T_n (where $(n,N)=1$) that act on modular forms.

Definition

If $f \in M_k(N,\chi)$ then $T_p(f) \in M_k(N,\chi)$ and its Fourier expansion is

$$(T_p(f))(z) = \sum_{n=0}^{\infty} (a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f))q^n.$$

That is,

$$a_n(T_p(f)) = a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f)$$

for $f \in M_k(N,\chi)$.

Nice properties of Hecke operators

In general T_n satisfies:

$$T_{ab} = T_a \cdot T_b \text{ if } \gcd(a,b) = 1$$

$$T_{p^r} = T_{p^{r-1}} \cdot T_p - \chi(p) \cdot p^{k-1} T_{p^{r-2}}$$

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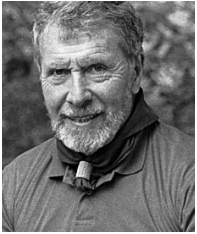
Definition

A modular form is an **eigenform** if for every prime p with $(p,N) = 1$, there exists a complex number λ_p such that

$$T_p(f) = \lambda_p \cdot f.$$

A Newform Theory for Cusp Forms



Newform Theory (Atkin-Lehner, 1970; Li, 1974):
$S_k(N, \chi)$ has a basis of special eigenforms called **newforms** of exact level M (where $M \mid N$) and their shifts by divisors d with $d \mid \frac{N}{M}$.

A Strong Multiplicity One Theorem for Cusp Forms

Theorem (Strong Multiplicity One)

If two (normalized) newforms $f \in S_k(N, \chi)$ and $g \in S_k(M, \chi)$ **have the same eigenvalues for all but finitely many of the Hecke operators (T_p)** *then $f = g$ and $M = N$.*

Note: Equivalently, the Fourier series have the same coefficients (a_p) for all but finitely many p.

A Stronger Multiplicity One Theorem for Cusp Forms

This has been improved over the years. The strongest known result for cusp forms:

Theorem (D. Ramakrishnan, 1994)

If two (normalized) newforms $f \in S_k(N, \chi)$ and $g \in S_k(M, \chi)$ **have the same p^{th} Hecke eigenvalues on a set of primes p with density** $> 7/8$*, then $f = g$ and $N = M$.*

A Newform Theory for Eisenstein series

There is an analogous newform theory for Eisenstein series:

Definition

Let N_1, N_2 be integers. Let χ_1, χ_2 be Dirichlet characters mod N_1, N_2 (respectively). For $k \geq 2$,

$$\sigma^{k-1}_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}.$$

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- There is an Eisenstein series $E_k(\chi_1, \chi_2)$ in the space $E_k(N_1N_2, \chi_1\chi_2)$ whose n^{th} Fourier coefficient for $n \geq 1$ is $\sigma^{k-1}_{\chi_1,\chi_2}(n)$.

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- When χ_1, χ_2 are primitive and $N_1N_2 = N$, then $E_k(\chi_1, \chi_2)$ is a newform of level N.

Strong Multiplicity One for Eisenstein Series

Theorem (Weisinger, 1971)

If f and g are normalized newforms in $E_k(N,\chi)$ and $E_k(M,\chi)$ (respectively) and if f and g **have the same eigenvalue of** T_p **for all but finitely many** p*, then $f = g$ and $M = N$.*

A Stronger Multiplicity One Theorem

Theorem (Linowitz + T., 2015)

If $f \in E_k(N, \chi_f)$ and $g \in E_{k'}(N', \chi_g)$ are newforms **whose** p^{th} **Hecke eigenvalues are equal for a set of primes with density** $> 1/2$, *then* $f = g, k = k', N = N'$ *and* $\chi_f = \chi_g$.

It turns out that our result is "best possible."

We can construct 2 newforms f, g whose Hecke eigenvalues agree for a set of primes p with density $1/2$:

Example

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$$\lambda_g = \psi(p) \cdot \lambda_f.$$

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So, f and g have the same eigenvalue whenever $\psi(p) = 1$, which happens $1/2$ of the time.

Sign changes

Let $f \in S_k(N, \chi)$ be a newform and let $\lambda_f(p)$ be the eigenvalue of f with respect to the Hecke operator T_p.

Theorem

There are infinitely many primes for which $\lambda_f(p) > 0$ and infinitely many for which $\lambda_f(p) < 0$.

The first sign change (cusp forms)




Theorem (Kowalski, Lau, Soundararajan, Wu, 2010)

For k even and $(n, N) = 1$, we have $\lambda_f(n) < 0$ for some $n \ll (k^2N)^{9/20}$.

An Improvement on the K-L-S-W Theorem

Theorem (Matomäki, 2012)

For k even and $(n, N) = 1$, we have $\lambda_f(n) < 0$ for some $n \ll (k^2N)^{\mathbf{3/8}}$.

An improvement on K-L-S-W for $N = 1$

Theorem (Matomäki & Radziwiłł, 2015)

For $N = 1$, there exists a positive constant δ such that, for x sufficiently large, the sequence $(\lambda_f(n))_{n \leq x}$ has at least δx sign changes. Furthermore, half of the non-zero $\lambda_f(n)$ are positive and half are negative.

The sign of $\sigma_{\chi_1,\chi_2}^{k-1}(n)$

Lemma (Linowitz + T., 2015)

If $(n, N) = 1$ and χ_1, χ_2 are quadratic, the sign of $\sigma_{\chi_1,\chi_2}^{k-1}(n)$ is completely determined by the behavior of $\chi_2(n)$.

Proof: From the definition of $\sigma_{\chi_1,\chi_2}^{k-1}(n)$, we have

$$\sigma_{\chi_1,\chi_2}^{k-1}(n) = \sum_{d|n} \chi_1(n/d)\chi_2(d)d^{k-1}$$

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The sign of $\sigma_{\chi_1,\chi_2}^{k-1}(n)$

Since $|\chi_1(d)|, |\chi_2(d)| \leq 1$ for all $d \in \mathbb{Z}^+$,

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The sign of $\sigma^{k-1}_{\chi_1,\chi_2}(n)$

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Thus, the $\chi_2(n)n^{k-1}$ term dominates and

$$\operatorname{sgn}\, \sigma^{k-1}_{\chi_1,\chi_2}(n) = \chi_2(n).$$

Two immediate corollaries

Corollary

For infinitely many n, we have $\sigma^{k-1}_{\chi_1,\chi_2}(n) > 0$. Similarly, $\sigma^{k-1}_{\chi_1,\chi_2}(n) < 0$ holds for infinitely many n.

Corollary

Let p_0 represent the positive integer where the first sign change of $\sigma^{k-1}_{\chi_1,\chi_2}(n)$ (from positive to negative) occurs. Then, for any fixed $\varepsilon > 0$, we have

$$p_0 \ll_\varepsilon N^{\frac{1}{4\sqrt{e}}+\varepsilon}.$$

Sequences of signs (cusp forms)

Theorem (Kowalski, Lau, Soundararajan, Wu, 2010)

Let N be squarefree and $k \geq 2$ an even integer. Let $\{\varepsilon_p\}$ be a sequence of signs. Let $H_k^*(N, \chi)$ be the set of all normalized newforms in $S_k(N, \chi)$, where χ is principal. For any ε with $0 < \varepsilon < 1/2$, there exists $c > 0$ such that

$$\frac{1}{|H_k^*(N,\chi)|}\#\{f \in H_k^*(N,\chi) : \lambda_f(p) = \varepsilon_p \text{ for all } p \leq x, p \nmid N\} \geq \left(\frac{1}{2} - \varepsilon\right)^{\pi(x)}$$

for $x = c\sqrt{(\log kN)(\log\log kN)}$, provided kN is large enough.

Sequences of signs (Eisenstein series)

Theorem (Linowitz + T., 2015)

Let $\mathcal{D} = \{(D_1, D_2) : |D_1 D_2| \leq x\}$. *Given a sequence of signs* $\varepsilon_i = \{0, \pm 1\}$, *we have*

$$\frac{1}{|\mathcal{D}|}\#\{(D_1, D_2) \in \mathcal{D} : \operatorname{sgn} \sigma^{k-1}_{\chi_1,\chi_2}(p_i) = \varepsilon_i, 1 \leq i \leq k\}$$

$$\to \Bigg(\prod_{\substack{\varepsilon_i = 0 \\ 1 \leq i \leq k}} \frac{1}{(p_i+1)^2}\Bigg)\Bigg(\prod_{\substack{\varepsilon_i \neq 0 \\ 1 \leq i \leq k}} \frac{p_i(p_i+2)}{2(p_i+1)^2}\Bigg),$$

as $x \to \infty$.

The first sign change on average

Conjecture (Linowitz + T., 2015)

Let $\eta(D_1, D_2)$ *represent the smallest* p_i *for which* $\operatorname{sgn}\,\sigma^{k-1}_{\chi_1,\chi_2}(p_i) = -1$. *As* $x \to \infty$, *one would expect*

$$\frac{\sum_{|D_1D_2| \leq x} \eta(D_1, D_2)}{\sum_{|D_1D_2| \leq x} 1} \to \theta,$$

where

$$\theta := \sum_{\ell=1}^{\infty} \frac{p_\ell^2(p_\ell + 2)}{2(p_\ell + 1)^2} \prod_{i=1}^{\ell-1} \frac{2 + p_i(p_i + 2)}{2(p_i + 1)^2}.$$

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Numerically,

$$\theta = 3.97502239026675398477347591051755102460193555\ldots.$$

Heuristic argument

Heuristically, one would expect

$$\frac{\sum_{|D_1D_2|\leq x}\eta(D_1,D_2)}{\sum_{|D_1D_2|\leq x}1}=\sum_{\ell=1}^{\infty}p_\ell\cdot\mathrm{Prob}\left(\eta(D_1,D_2)=p_\ell\right)$$

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where the final equality follows from our theorem.

Thank you!