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Reviews

Reviewed by Lola Thompson

REVIEWS

Edited by **Darren Glass**
Department of Mathematics, Gettysburg College, Gettysburg, PA 17325

An Introduction to the Circle Method. By M. Ram Murty & Kaneenika Sinha, American Mathematical Society, 2023. 258 pp., ISBN 978-1-4704-7203-0, $59.00.

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When I was first approached to review a book on the circle method aimed at undergraduates, I was a bit skeptical. After all, this is a subject that normally gets covered in the latter portion of a semester-long master's-level course in analytic number theory, or possibly in a more-advanced "topics" course aimed at Ph.D. students. Surely it would be too specialized for students in most bachelor's programs! However, when I saw that Ram Murty was one of the authors, I felt more optimistic. Murty is a co-author (along with Alina Cojocaru) of the sieve methods book that I most often recommend to students who are learning the subject for the first time [1]. He is a gifted expositor who really understands how to pitch ideas at the right level for this audience. I was curious to see what he and Kaneenika Sinha could pull off with another notoriously impenetrable topic. I spent the next few weeks eagerly awaiting the mail delivery, even paying the hefty ransom that the Dutch customs agents demand for non-EU mail because I was very curious about this book.

When I finally received *An Introduction to the Circle Method*, I was immediately struck by the amount of prior background in number theory that the authors demand from their readers: zero. That said, while the authors go all the way back to defining the principle of mathematical induction and stating the division algorithm, a reader with some familiarity with elementary number theory will be much better-equipped to read this book (and such a person will then be able to skip Chapters 2–4). While undergraduates could, in theory, learn elementary number theory along the way, it will be easier for students who have already internalized the main ideas from these early chapters.

It is important to note that, while a background in number theory is not assumed, the authors do take for granted that the reader has seen some complex analysis. From Chapter 5 onwards, the reader will want to have some facility with contour integration. Otherwise, much of the rest of the book will feel like a series of black boxes. That said, for students with a solid background in both real and complex analysis, this book would make a good textbook for a very specialized one or two semester course (depending on whether elementary number theory is a pre-requisite). It would also be a natural choice for an independent reading course for an extremely motivated student.

Before I go any further, it is important to give a brief explanation of the circle method. We can express certain additive problems in number theory using *exponential sums*. For example, suppose that we want to count how many s-tuples of nonnegative integers $r_1, \dots, r_s$ satisfy

$$r_1 + r_2 + \cdots + r_s = n,$$

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for some fixed positive integer n. It turns out that it is straightforward to show that the number of such s-tuples is equal to the integral

$$\int_0^1 f^s(\alpha)e(-n\alpha)d\alpha \tag{1}$$

where $e(\alpha) := e^{2\pi i\alpha}$ and $f^s(\alpha) = \sum_{r_1,\ldots,r_s\le n} e((r_1+\cdots+r_s)\alpha)$. In particular, there is at least one such s-tuple if and only if the integral is positive. As α ranges from 0 to 1, $e(\alpha)$ moves around the unit circle—hence the name "the circle method." In many applications of the circle method, it is useful to observe that the function $f(\alpha) = \sum_{k\le n} e(k\alpha)$ assumes atypically large values at rational numbers $\alpha = a/q$ with small denominators q. Thus it makes sense to separate the interval [0, 1] into two parts: the so-called *major arcs*, which are unions of small intervals surrounding rational numbers with small denominators, and the *minor arcs*, which are unions of everything else in [0, 1]. Then, one uses tools from asymptotic analysis to estimate the integral along the major and minor arcs separately. In an ideal world, the integral along the major arcs can be estimated fairly precisely, and the integral along the minor arcs can be shown to be smaller than the estimate along the major arcs (and, thus, it serves as an error term). This is, of course, explained in much greater detail by Murty and Sinha in Chapters 6–10.

One of the book's strengths is that it spells out the steps that are typically omitted from other books on the circle method. Steps that are "clear" and proofs that are "left to the reader" in other sources are written out in full, glorious detail here. For that reason, even experts who are assigned to teach a course in the circle method but who do not have a great deal of time on their hands may appreciate having this book on their shelves as a secret reference. (Shhh, don't tell my students that I wrote this!)

Many courses in number theory aim for breadth, but Murty and Sinha take the opposite approach: they provide precisely what is needed, and no more, in order to build a narrow path that ultimately leads to attacking two important problems in additive number theory: Waring's Problem and Goldbach's Conjecture. Waring's problem asks whether every natural number can be written as a sum of at most s kth powers. Goldbach's Conjecture is the statement that every even integer greater than 2 can be written as a sum of two prime numbers. From the discussion above, this is equivalent to saying that, for any even $n > 2$,

$$\int_0^1 f^2(\alpha)e(-n\alpha)d\alpha > 0,$$

where $f(\alpha) = \sum_{p\le n} e(\alpha p)$. Of course, Goldbach's Conjecture is still an open problem, and there are very good reasons to believe that the circle method alone cannot be used to prove Goldbach's conjecture. The authors give the same explanation as Davenport [2] for why the Circle Method has thus far failed to prove Goldbach's Conjecture. Namely, when we plug the Goldbach problem into circle method machinery, suddenly the minor arcs are too large to serve as an error term; instead, they contribute to the main term.

This perhaps leaves the reader with the impression that if we were to try just a little bit harder, we might be able to partition the unit interval in another way so that there is a main term that we can estimate, and a genuine error term. However, the obstruction is more serious than what is mentioned by Murty and Sinha. Namely, when we attack the Goldbach problem by means of the circle method, the exponent in (1) is 2. If the exponent were 3 then we could bound the integral by the square of the L^2-norm of f times a factor of $\min_{\alpha\in\mathfrak{m}} |f(\alpha)|$, with $\mathfrak{m}$ denoting the set of minor arcs;

we can then determine the L^2 norm easily. However, that trick is not available and we do not have a good bound on the L^1 norm. As Terry Tao remarks in his blog [4], giving asymptotics on the minor arcs is not any easier than proving Goldbach itself. He proclaims that "tight bounds on minor arc exponential sums are basically just a Fourier reformulation of the underlying binary problems being considered." In other words, it's not just that one cannot prove Goldbach in quite the way that Murty and Sinha rightly say does not work; the contribution of the "minor" arcs is no longer minor, but actually part of the main term. This leaves us completely stuck because we do not have asymptotics on the minor arcs. It is curious that the word "Fourier" does not appear anywhere in Murty and Sinha's book—this is presumably a choice that was made in order to avoid overburdening the undergraduate reader with yet another analysis prerequisite (one that is not often part of a bachelor's curriculum in mathematics). This is an understandable choice, but it does leave the reader with only part of the story.

At this point, you may be wondering: why place so much emphasis on a problem that is still open and unlikely to be solvable using the method that is the focus of this book? The answer is that there is a related problem, called the Ternary Goldbach Theorem, that was proved in several stages using the circle method, with a full solution finally given in 2013. The Ternary Goldbach Theorem states that every odd number greater than 5 can be written as a sum of three primes. In this case, the exponent in Equation (1) is 3 and we have the tools described above at our disposal. The final pieces of the Ternary Goldbach proof, due to Harald Helfgott, use extremely delicate estimates that go beyond the scope of this book. Murty and Sinha prove Ivan Vinogradov's classical result, namely, that the Ternary Goldbach Theorem holds for all sufficiently large odd numbers. (Vinogradov himself was building on the work of G. H. Hardy and John Littlewood, who had proved the same result conditionally on the generalized Riemann hypothesis. Vinogradov's great achievement was to give meaningful bounds on the minor arcs, unconditionally.) Murty and Sinha take a modern approach to Vinogradov's result, incorporating some improvements due to Vaughan.

At the end of the "preparatory chapters," the authors provide a nice summary of the sequence of results that led to the full proof of the Ternary Goldbach conjecture. We feel the suspense as Vinogradov's "sufficiently large" result is made effective, and as Borodzkin's bound of $10^{4008659}$ is reduced to Helfgott's bound of 10^{27}, and separately computer-verified (by Helfgott and David Platt [3]) for odd n up to about 10^{30}. Mathematics students often complain that the topics that they study were all figured out hundreds of years ago, but here we see recent research in action! As unglamorous as it may feel to reduce one large power of 10 to another, it's pretty awe-inspiring to stand back and take in the contributions of so many mathematicians over the years.

Another strong suit of this book is that it is peppered with interesting historical anecdotes throughout. The "preparatory chapters" do not merely introduce notation and basic concepts; they also tell the stories of the mathematicians who pioneered the use of the circle method. We learn that, even though the circle method is frequently called the "Hardy-Littlewood circle method," it may have already existed in a rudimentary form in Ramanujan's mind prior to his famous visit to Hardy at Cambridge (what is certainly correct is that it first appeared in a paper by Hardy and Ramanujan; the book's assertion that it originated with Ramanujan is up for debate, though he certainly played a role in its origins—indeed, the appellation "Hardy-Ramanujan-Littlewood method" is becoming more common). While the method was originally developed to find asymptotics for the partition function, its full power was later revealed when Hardy and Littlewood figured out applications such as Waring's problem.

One possible criticism of this book is that it uses, as motivation, the same problem that is highlighted in nearly every introductory lecture on the circle method: Waring's

Problem. I spent some time trying to envision a different low-hanging problem that would also provide a natural introduction to the full power of the circle method and I'm really struggling to find a reasonable alternative. Perhaps I am just unimaginative, or perhaps there is a reason why every "Circle Method 101" lecture starts with Waring's problem.

Another possible criticism is that certain parts of the book may not age well. The section called "The future of the circle method" reads a bit like an Oberwolfach report. It lists many of the results that we currently view as important (and the people that we currently view as influential), but one cannot ignore the fact that there are likely important applications of the circle method being developed right now with impacts that won't be known for years. Analytic number theory is developing rapidly, with numerous seemingly-impossible problems startlingly solved in the last decade. For example, Fields Medals have been awarded to analytic number theorists at 4 out of the last 5 International Congresses of Mathematicians. All of this is to say that "The future of the circle method" may well feel like Disney's Tomorrowland in thirty years: a quaint snapshot into the mindset of analytic number theorists in the early 2020s and their retro-futuristic ideas of where the field is heading.

In spite of these mild criticisms, I overall applaud Murty and Sinha for making the circle method accessible to a wider audience. I strongly suspect that *An Intoduction to the Circle Method* will enter the regular rotation of analytic number theory books that I recommend to my students (indeed, I have already recommended it to one student). My only hope is that the Dutch customs agents won't detain their copies.

ORCID
Lola Thompson http://orcid.org/0000-0001-6692-4108

REFERENCES

[1] Cojocaru, A., Murty, R. (2005). *An Introduction to Sieve Methods and Their Applications*. Cambridge: Cambridge University Press.
[2] Davenport, H. (2000). *Multiplicative Number Theory*. New York: Springer-Verlag.
[3] Helfgott, H., Platt, D. (2013) Numerical verification of the Ternary Goldbach conjecture up to $8.875 \cdot 10^{30}$. *Exp. Math.* 22(4): 406–409.
[4] Tao, T. (2012). *Heuristic Limitations of the Circle Method.* Terence Tao's blog.

Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD Utrecht
l.thompson@uu.nl