# PRACTICAL PRETENDERS

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ABSTRACT. Following Srinivasan, an integer $n \geq 1$ is called *practical* if every natural number in $[1, n]$ can be written as a sum of distinct divisors of $n$. This motivates us to define $f(n)$ as the largest integer with the property that all of $1, 2, 3, \ldots, f(n)$ can be written as a sum of distinct divisors of $n$. (Thus, $n$ is practical precisely when $f(n) \geq n$.) We think of $f(n)$ as measuring the "practicality" of $n$; large values of $f$ correspond to numbers $n$ which we term *practical pretenders*. Our first theorem describes the distribution of these impostors: Uniformly for $4 \leq y \leq x$,

$$\#\{n \leq x : f(n) \geq y\} \asymp \frac{x}{\log y}.$$

This generalizes Saias's result that the count of practical numbers in $[1, x]$ is $\asymp \frac{x}{\log x}$.

Next, we investigate the maximal order of $f$ when restricted to non-practical inputs. Strengthening a theorem of Hausman and Shapiro, we show that every $n > 3$ for which

$$f(n) \geq \sqrt{e^{\gamma} n \log \log n}$$

is a practical number.

Finally, we study the range of $f$. Call a number $m$ belonging to the range of $f$ an *additive endpoint*. We show that for each fixed $A > 0$ and $\epsilon > 0$, the number of additive endpoints in $[1, x]$ is eventually smaller than $x/(\log x)^A$ but larger than $x^{1-\epsilon}$.

## 1. INTRODUCTION

In 1948, Srinivasan [15] initiated the study of *practical numbers*, natural numbers $n$ with the property that each of $1, 2, 3, \ldots, n-1$ admits an expression as a sum of distinct divisors of $n$. For example, every power of 2 is practical (since every natural number admits a binary expansion), but there are many unrelated examples, such as $n = 6$ or $n = 150$. Srinivasan posed two problems: Classify all practical numbers and say something interesting about their distribution.

The first of these tasks was carried to completion by Stewart [16] in 1954. The same classification was discovered independently, and almost concurrently, by Sierpiński [14]. Given a natural number $n$, write its canonical prime factorization in the form

$$(1.1) \qquad n := p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, \quad \text{where} \quad p_1 < p_2 < \cdots < p_r.$$

Put $n_0 = 1$, and for $1 \leq j \leq r$, put $n_j := \prod_{i=1}^{j} p_i^{e_i}$. Using $\sigma$ for the usual sum-of-divisors function (so that $\sigma(m) := \sum_{d|m} d$), the number $n$ is practical if and only if

$$(1.2) \qquad p_{j+1} \leq \sigma(n_j) + 1 \quad \text{for all} \quad 0 \leq j < r.$$

Below, we refer to this as the *Stewart–Sierpiński classification* of practical numbers. This criterion implies, in particular, that all practical numbers $n > 1$ are even. Stewart and Sierpiński also showed that if all of the inequalities (1.2) hold, then not only are all integers in $[1, n-1]$ expressible as a sum of distinct divisors of $n$, but the same holds for all integers in the longer interval $[1, \sigma(n)]$. Note that $[1, \sigma(n)]$ is the largest interval one could hope to represent, since the sum of all distinct divisors of $n$ is $\sigma(n)$.

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The distribution of practical numbers has proved more recalcitrant. Let $P(x)$ denote the count of practical numbers not exceeding $x$. Already in 1950, Erdős [2] claimed he could show that the practical numbers have asymptotic density zero, i.e., that $P(x) = o(x)$ as $x \to \infty$, but he gave no details. In 1984, Hausman and Shapiro [6] made the more precise assertion that $P(x) \leq x/(\log x)^{\beta+o(1)}$, with $\beta = \frac{1}{2}(1 - 1/\log 2)^2 \approx 0.0979\ldots$. Their proof has an error (specifically, [6, Lemma 3.2] is incorrect); one should replace $\beta$ with the smaller exponent $1 - \frac{1+\log\log 2}{\log 2} \approx 0.0860713$. Much sharper results on $P(x)$ were soon established by Tenenbaum [17, 19], who proved that $P(x) = \frac{x}{\log x}(\log\log x)^{O(1)}$. By a refinement of Tenenbaum's methods, Saias [13] established in 1997 what is still the sharpest known result: There are absolute constants $c_1$ and $c_2$ with

$$(1.3) \qquad c_1 \frac{x}{\log x} \leq P(x) \leq c_2 \frac{x}{\log x} \quad \text{for all} \quad x \geq 2.$$

On the basis of the numerical data, Margenstern [9] has conjectured that $\frac{P(x)}{x/\log x}$ tends to a limit $\approx 1.341$.

In this paper, we are concerned with what we term *near-practical numbers* or *practical pretenders*. Define $f(n)$ as the largest integer with the property that all of the numbers $1, 2, 3, \ldots, f(n)$ can be written as a sum of distinct divisors of $n$. By definition, $n$ is practical precisely when $f(n) \geq n - 1$. We define a *near-practical* number as one for which $f(n)$ is "large". This definition is purposely vague; its nebulous nature suggests that we investigate the behavior of the two-parameter function

$$N(x, y) := \#\{n \leq x : f(n) \geq y\}$$

for all $x$ and $y$. Our first result gives the order of magnitude of the near-practical numbers for essentially all interesting choices of $x$ and $y$.

**Theorem 1.1.** *There are absolute positive constants $c_3$ and $c_4$ so that for $4 \leq y \leq x$, we have*

$$c_3 \frac{x}{\log y} \leq N(x, y) \leq c_4 \frac{x}{\log y}.$$

**Remark.** To see why the technical restriction $y \geq 4$ is necessary, note that $N(x, x) = 0$ for all $3 < x < 4$.

Theorem 1.1 has the following easy corollary, proved in §3.

**Corollary 1.2.** *For each $m$, the set of natural numbers $n$ with $f(n) = m$ possesses an asymptotic density, say $\rho_m$. The constant $\rho_m$ is positive whenever there is at least one $n$ with $f(n) = m$. Moreover,*

$$\sum_{m=1}^{\infty} \rho_m = 1.$$

We call a natural number $m$ for which $\rho_m$ is nonvanishing (equivalently, an $m$ in the image of $f$) an *additive endpoint*. Thus, Corollary 1.2 shows that $\rho_m$ is the probability mass function for additive endpoints. The first several additive endpoints are

$$1, 3, 7, 12, 15, 28, 31, 39, 42, 56, 60, 63, 73, 90, 91, 96, 100, 104, 108, 112, 120, \ldots.$$

Just from this limited data, one might conjecture that $\rho_m$ is usually zero, i.e., zero apart from of a set of $m$ of vanishing asymptotic density. This guess is confirmed, in a much sharper form, in our next theorem.

**Theorem 1.3.** *For each fixed $A > 0$ and all $x \geq 3$, the number of integers in $[1, x]$ which occur as additive endpoints is $\ll_A x/(\log x)^A$. In the opposite direction, the number of additive endpoints up to $x$ exceeds*

$$x/\exp(c_5(\log\log x)^3)$$

*for all large $x$, for some absolute constant $c_5 > 0$.*

Above, we noted Stewart's result that if $f(n) \geq n - 1$, then $f(n) = \sigma(n)$. In this statement, a weak lower estimate on $f(n)$ implies that $f(n)$ is as large as possible. Hausman and Shapiro [6] proposed investigating the extent of this curious phenomenon. More specifically, they asked for the slowest-growing monotone function $g(n)$ for which $f(n) \geq g(n)$ implies (at least for $n$ large) that $n$ is practical. Set

$$H(n) := \sqrt{e^\gamma n \log\log n},$$

where $\gamma$ is the Euler–Mascheroni constant. The next proposition appears as [6, Theorems 2.1, 2.2].

**Proposition 1.4.** *Let $\epsilon > 0$. Apart from finitely many exceptional $n$, all solutions to $f(n) \geq (1 + \epsilon)H(n)$ are practical. On the other hand, there are infinitely many non-practical $n$ with $f(n) \geq (1 - \epsilon)H(n)$.*

Our final theorem removes the factor $1 + \epsilon$ from the first half of Proposition 1.4.

**Theorem 1.5.** *If $n > 3$ and $f(n) \geq H(n)$, then $n$ is practical.*

**Notation.** We use the Landau–Bachmann $o$ and $O$ symbols, as well as Vinogradov's $\ll$ notation, with their usual meanings; subscripts indicate dependence of implied constants. We write $\omega(n) := \sum_{p|n} 1$ for the number of distinct prime factors of $n$ and $\Omega(n) := \sum_{p^k|n} 1$ for the number of prime factors of $n$ counted with multiplicity; $\Omega(n; y) := \sum_{p^k|n,\ p\leq y} 1$ denotes the number of prime divisors of $n$ not exceeding $y$, again counted with multiplicity. The number of divisors of $n$ is denoted $d(n)$. We use $P^-(m)$ for the smallest prime factor of $m$, with the convention that $P^{-1}(1)$ is infinite. Absolute positive constants are denoted by $c_1, c_2, c_3$, etc., and have the same meaning each time they appear.

## 2. Proofs of Theorem 1.1 and Corollary 1.2

We begin by recording some useful lemmas. Our first gives a formula for $f(n)$ in terms of the prime factorization of $n$.

We assume that the factorization of $n$ has been given in the form (1.1). We define $n_0 := 1$ and $n_j := \prod_{1\leq i\leq j} p_i^{e_i}$. Let $0 \leq j < r$ be the first index for which $p_{j+1} > \sigma(n_j) + 1$, putting $j = r$ if no such index exists (i.e., if $n$ is practical). Then $n_j$ is a practical number, by the Stewart–Sierpiński classification, and we call $n_j$ the *practical component* of $n$.

**Lemma 2.1.** *We have $f(n) = \sigma(n_j)$, where $n_j$ is the practical component of $n$.*

*Proof.* Since $n_j$ is practical, $f(n) \geq f(n_j) = \sigma(n_j)$. On the other hand, $\sigma(n_j) + 1$ is not representable as a sum of proper divisors of $n$. Indeed, if $d$ is a divisor of $n$ involved in an additive representation of $\sigma(n_j) + 1$, then $d \leq \sigma(n_j) + 1 < p_{j+1} < p_{j+2} < \cdots < p_r$. It follows that the only primes dividing $d$ are $p_1, \ldots, p_j$, so that $d$ is a divisor of $n_j$. But the largest number which can be formed as a sum of distinct divisors of $n_j$ is $\sigma(n_j)$, which is smaller than $\sigma(n_j) + 1$. So $\sigma(n_j) + 1$ is not representable as a sum of distinct divisors of $n$, and hence $f(n) = \sigma(n_j)$, as claimed. $\qquad\square$

The following lemma was observed by Margenstern [9, Corollaire 1] to follow from the Stewart–Sierpiński classification.

**Lemma 2.2.** *If $n$ is practical and $m \leq \sigma(n) + 1$, then $mn$ is practical.*

We now employ Lemma 2.2 to show that reasonably short intervals contain a positive proportion of practical numbers.

**Lemma 2.3.** *Let $\epsilon > 0$. For $x > x_0(\epsilon)$, the number of practical numbers in $((1 - \epsilon)x, x]$ is $\gg_\epsilon x/\log x$.*

*Proof.* We can assume that $0 < \epsilon < 1$. With $c_1$ and $c_2$ as defined in (1.3), we set $r := \lceil 2c_2/c_1 \rceil$ and $s := \lceil 1/\epsilon \rceil$. From (1.3), we have that for large $x$ (depending on $\epsilon$), the number of practical numbers in the interval $(x/rs, x/s]$ is

$$\geq c_1 \frac{x/s}{\log(x/s)} - c_2 \frac{x/rs}{\log(x/rs)} > \frac{c_1}{3s} \frac{x}{\log x} \geq \frac{c_1}{6} \epsilon \frac{x}{\log x}.$$

By the pigeonhole principle, one of the intervals $(\frac{x}{s+1}, \frac{x}{s}], (\frac{x}{s+2}, \frac{x}{s+1}], \ldots, (\frac{x}{rs}, \frac{x}{rs-1}]$ contains $> \frac{c_1}{6rs} \epsilon x/\log x \gg \epsilon^2 x/\log x$ practical numbers. Suppose this interval is $(\frac{x}{j+1}, \frac{x}{j}]$, where $s \leq j < rs$, and let $n$ be a practical number contained within. If $x > (rs)^2$, then $j < \frac{x}{j+1} < n$, and so $jn$ is practical by Lemma 2.2. (Note that the lower bound on $x$ assumed here depends only on $\epsilon$.) Letting $n$ run through the practical numbers in $(\frac{x}{j+1}, \frac{x}{j}]$, we obtain $\gg \epsilon^2 x/\log x$ practical numbers $jn \in (x\frac{j}{j+1}, x]$. But $(x\frac{j}{j+1}, x] \subset ((1 - \epsilon)x, x]$, by our choice of $s$. This proves Lemma 2.3. Moreover, we have shown that the implied constant in the lemma statement may be taken proportional to $\epsilon^2$. $\qquad\square$

The next result, due to Hausman and Shapiro [6, Theorem 4.1], shows that substantially shorter intervals than those considered in Lemma 2.3 always contain at least one practical number.

**Lemma 2.4.** *For all real $x \geq 1$, there is a practical number $x < n < x + 2x^{1/2}$.*

Let $\Phi(x, y)$ denote the number of natural numbers $n \leq x$ divisible by no primes $\leq y$. The following lemma is a consequence of Brun's sieve. Variants can be found, e.g., as [3, Theorem 1, p. 201] or [18, Theorem 3, p. 400].

**Lemma 2.5.** *Uniformly for $2 \leq y \leq x$, we have $\Phi(x, y) \ll x/\log y$. If we assume also that $x > c_6 y$ for a suitable large absolute constant $c_6$, then $\Phi(x, y) \gg x/\log y$.*

We now prove Theorem 1.1, treating the upper and lower estimates separately.

*Proof of the upper bound in Theorem 1.1.* Suppose that $n \leq x$ and $f(n) \geq y$. By the upper bound in (1.3), we may restrict our attention to non-practical $n$. Let $d$ be the practical component of $n$ and write $n = dq$. By Lemma 2.1, $\sigma(d) = f(n)$. In particular, since we are assuming that $f(n) \geq y \geq 4$, we must have that $d > 1$. Moreover, since $n$ is not practical, $d < n$. Thus, $q > 1$ and

$$P^-(q) > \sigma(d) + 1 > d.$$

Hence,

$$d^2 < d \cdot P^-(q) \leq dq = n \leq x,$$

and so $d \leq \sqrt{x}$.

Given $d$, the number of possibilities for $n$ is bounded above by the number of $q \leq x/d$ with $P^-(q) > d$. Since $2 \leq d \leq x/d$, we may apply Lemma 2.5 to find that the number of possibilities for $q$ is $\ll \frac{x}{d\log d}$. Since $\sigma(d) = f(n) \geq y$ and (crudely) $\sigma(d) < d^2$, it

follows that $d > \sqrt{y}$. Hence, using partial summation and (1.3), we see that the number of possibilities for $n$ is

$$
\ll x \sum_{\substack{\sqrt{y} < d \le \sqrt{x} \\ d \text{ practical}}} \frac{1}{d \log d} \le x \frac{P(\sqrt{x})}{\sqrt{x} \log \sqrt{x}} + x \int_{\sqrt{y}}^{\sqrt{x}} P(t) \frac{1 + \log t}{(t \log t)^2} \, dt
$$

$$
\ll \frac{x}{(\log x)^2} + x \int_{\sqrt{y}}^{\sqrt{x}} \frac{dt}{t (\log t)^2} \ll \frac{x}{(\log x)^2} + \frac{x}{\log y} \ll \frac{x}{\log y}. \qquad \square
$$

*Proof of the lower bound in Theorem 1.1.* The proof is suggested by that offered for the upper bound, but some care is necessary to ensure uniformity throughout the stated range of $x$ and $y$.

First, we treat the range when $x^{1/10} \le y \le x$. In this domain, we use the trivial lower bound $N(x, y) \ge N(x, x)$. We estimate the right-hand side from below by counting practical numbers $n$ belonging to the interval $[\frac{x+1}{2}, x]$. Note that for such $n$, we have $f(n) = \sigma(n) \ge 2n - 1 \ge x$ (using for the first inequality that $n - 1$ is a sum of proper divisors of $n$), and so $n$ is indeed counted by $N(x, x)$.

If $6 \le x \le 11$, then $n = 6$ is a practical number in $[\frac{x+1}{2}, x]$. Similarly, if $4 \le x \le 6$, then $n = 4$ works. Finally, if $x \ge 11$, then Lemma 2.4 gives a practical number $n$ with

$$
\frac{x+1}{2} < n < \frac{x+1}{2} + 2\sqrt{\frac{x+1}{2}} \le x.
$$

Hence, we always have $N(x, x) \ge 1$. (Recall that we only consider $x \ge 4$.) Moreover, by Lemma 2.3, there are $\gg x/\log x$ practical numbers in $[\frac{x+1}{2}, x]$ once $x$ is large. It follows that $N(x, x) \gg x/\log x$ for all $x \ge 4$. So if $x^{1/10} \le y \le x$, then

$$
N(x, y) \ge N(x, x) \gg x/\log x \gg x/\log y,
$$

which gives the lower bound of the theorem in this case.

Now suppose that $4 \le y < x^{1/10}$. We consider numbers of the form $n = dq \le x$, where $d$ is a practical number in $(y, y^3]$ and where $P^-(q) > y^6$. For any such $n$, we have $f(n) \ge f(d) \ge d > y$. Moreover, each $n$ constructed in this way arises exactly once, since $q$ is determined as the largest divisor of $n$ supported on primes $> y^6$. Given $d$, the number of corresponding $q$ is $\Phi(x/d, y^6)$. If $x$ is large, then

$$
\frac{x/d}{y^6} \ge \frac{x}{y^9} \ge x^{1/10} > c_6,
$$

and so Lemma 2.5 gives

$$
(2.1) \qquad \Phi(x/d, y^6) \gg \frac{x}{d \log y}.
$$

On the other hand, (2.1) is trivial for bounded $x$, since 1 is always counted by $\Phi(x/d, y^6)$. Thus, (2.1) holds in any case. Hence, the number of $n$ constructed in this way is

$$
\gg \frac{x}{\log y} \sum_{\substack{y < d \le y^3 \\ d \text{ practical}}} \frac{1}{d}.
$$

That the sum appearing here is $\gg 1$ for large $y$ follows from partial summation and the lower bound in (1.3). For bounded $y$, the sum is also $\gg 1$, since Lemma 2.4 guarantees that there is at least one practical number between $y$ and $y^3$. (Certainly $y + 2y^{1/2} < 3y < y^3$ when $y \ge 4$.) This completes the proof of the lower bound. $\qquad \square$

*Proof of Corollary 1.2.* One can detect whether or not $f(n) = m$ given just the list of divisors of $n$ not exceeding $m + 1$. Thus, whether or not $f(n) = m$ depends only on the residue class of $n$ modulo $(m+1)!$. This gives the first two assertions of the corollary. For the third, notice that $1 - \sum_{n=1}^{N} \rho_m$ represents the density of the set of $n$ with $f(n) > N$, which is $\ll 1/\log N$ by Theorem 1.1. Letting $N \to \infty$ completes the proof.          $\square$

## 3. PROOF OF THEOREM 1.3

We divide the proof of Theorem 1.3 into two parts.

3.1. **The upper bound in Theorem 1.3.** Central to the proof of both halves of Theorem 1.3 is the observation, immediate from Lemma 2.1, that $m$ belongs to the range of $f$ precisely when $m = \sigma(n)$ for some practical number $n$. Thus, we are really asking in Theorem 1.3 for estimates on the range of $\sigma$ restricted to practical inputs.

**Lemma 3.1.** *Let $A \geq 30$. Suppose that $x \geq 3$. If $n$ is a practical number with $x^{3/4} < n \leq x$, then either*

$$(3.1) \qquad\qquad\qquad\qquad \Omega(n) > 2A \log \log x$$

*or*

$$(3.2) \qquad\qquad\qquad\qquad \omega(n) > \frac{1}{2 \log A} \log \log x.$$

*Proof.* Since $n$ is practical, every integer in $[1, n]$ can be written as a subset-sum of divisors of $n$. Thus, $2^{d(n)} \geq n$, so we can use the hypothesis that $n > x^{3/4}$ to show

$$d(n) \geq \frac{\log n}{\log 2} > \frac{3/4}{\log 2} \log x > \log x.$$

Suppose that $n = \prod_{i=1}^{\ell} p_i^{e_i}$ is the factorization of $n$ into primes, where $\ell = \omega(n)$. Since $d(n) = \prod_{i=1}^{\ell}(e_i + 1) > \log x$, the inequality between the arithmetic and geometric means gives that

$$(3.3) \qquad\qquad \frac{1}{\ell^\ell} \left( \sum_{i=1}^{\ell} (e_i + 1) \right)^\ell \geq \prod_{i=1}^{\ell} (e_i + 1) > \log x.$$

Now assume that (3.1) fails. Then $\sum_{i=1}^{\ell}(e_i + 1) \leq 2 \sum_{i=1}^{\ell} e_i \leq 4A \log \log x$, and (3.3) gives $(\frac{4A \log \log x}{\ell})^\ell > \log x$. Writing $\ell = \lambda \log \log x$, we deduce that

$$\left( \frac{4A}{\lambda} \right)^{\lambda \log \log x} > \log x, \quad \text{and so} \quad \lambda \log \frac{4A}{\lambda} > 1.$$

This latter inequality, along with the condition $A \geq 30$, implies that $\lambda > \frac{1}{2 \log A}$ (by a short exercise in calculus). Since $\omega(n) = \lambda \log \log x$, we have (3.2).          $\square$

The next lemma, which belongs to the study of the anatomy of integers, bounds from above the number of $n$ with an abnormally large number of small prime factors.

**Lemma 3.2.** *Let $x, y \geq 2$, and let $k \geq 1$. The number of $n \leq x$ with $\Omega(n; y) \geq k$ is $\ll \frac{k}{2^k} x \log y$.*

**Remark.** As a special case (when $y = x$), the number of $n \leq x$ with $\Omega(n) \geq k$ is $\ll \frac{k}{2^k} x \log x$.

*Proof.* The proof is almost identical to that suggested in Exercise 05 of [4, p. 12], details of which can be found in [8, Lemmas 12, 13]. Thus, we only sketch it. Let $v := 2 - 1/k$. Let $g$ be the arithmetic function determined through the convolution identity $v^{\Omega(n;y)} = \sum_{d|n} g(d)$. Then $g$ is multiplicative. For $e \geq 1$, we have $g(p^e) = v^e - v^{e-1}$ if $p \leq y$, and $g(p^e) = 0$ if $p > y$. Hence,

$$\sum_{n \leq x} v^{\Omega(n;y)} = \sum_{d \leq x} g(d) \left\lfloor \frac{x}{d} \right\rfloor \leq x \sum_{d \leq x} \frac{g(d)}{d}$$

$$\leq x \prod_{p \leq y} \left( 1 + \frac{v-1}{p} + \frac{v^2 - v}{p^2} + \dots \right) = \frac{x}{2-v} \prod_{3 \leq p \leq y} \left( 1 + \frac{v-1}{p-v} \right).$$

Now $2 - v = 1/k$, and the rightmost product is

$$\leq \exp\left( \sum_{3 \leq p \leq y} \frac{v-1}{p-v} \right) \leq \exp\left( \sum_{3 \leq p \leq y} \frac{1}{p-2} \right) \leq \exp\left( \sum_{p \leq y} \frac{1}{p} + O(1) \right) \ll \log y.$$

Collecting our estimates, we have shown that

$$\sum_{n \leq x} v^{\Omega(n;y)} \ll kx \log y.$$

But each term with $\Omega(n;y) \geq k$ makes a contribution to the left-hand side that is $\geq v^k = (2 - 1/k)^k = 2^k(1 - \frac{1}{2k})^k \gg 2^k$. Thus, the number of such terms is $\ll \frac{k}{2^k} x \log y$. $\square$

The next lemma is a partial shifted-primes analogue of the Hardy-Ramanujan inequalities. A proof can be found in the text of Prachar [11, Lemma 7.1, p. 166] (cf. Erdős [1]). There a slightly stronger assertion is shown for shifted primes $p - 1$; only trivial changes are required to replace $p - 1$ with $p + 1$.

**Lemma 3.3.** *Let $t \geq 3$, and let $k \geq 1$. The number of primes $p \leq t$ with $\omega(p+1) = k$ is*

$$\ll \frac{t}{(\log t)^2} \left( \frac{(\log\log t + c_7)^{k+2}}{(k-1)!} + 1 \right).$$

*Proof of the upper bound in Theorem 1.3.* It is enough to prove the result for large values of $A$. Suppose that $m \leq x$ is an additive endpoint, and write $m = \sigma(n)$ with $n$ practical. Put $Z := 2A \log\log x$. The number of values of $m$ corresponding to an integer $n \leq x^{3/4}$ or an $n$ with $\Omega(n) > Z$ is, by Lemma 3.2,

$$\ll x^{3/4} + \frac{Z}{2^Z} x \log x \ll_A \frac{x}{(\log x)^A}.$$

Thus, with

$$Z' := \frac{1}{2 \log A} \log\log x,$$

Lemma 3.1 allows us to assume that

(3.4) $$\omega(n) \geq Z'.$$

We now show that most of the primes dividing $n$ make a large contribution to $\Omega(\sigma(n)) = \Omega(m)$. We claim we can assume that both of the following hold:

(i) There are fewer than $Z'/4$ primes $p$ for which $p^2 \mid n$.
(ii) There are fewer than $Z'/4$ primes $p$ dividing $n$ for which

(3.5) $$\Omega(p+1) \leq 16A \log A.$$

With $K := \lceil Z'/4 \rceil$, the number of $n \leq x$ which are exceptions to (i) is, by the multinomial theorem,

$$\leq x \sum_{\substack{d \leq x, \text{ squarefree} \\ \omega(d) = K}} \frac{1}{d^2} \leq \frac{x}{K!} \left( \sum_{p \leq x} \frac{1}{p^2} \right)^K \leq x(e/K)^K < x/(\log x)^A,$$

once $x$ is large. (We use here that $\sum p^{-2} < 1$ and the elementary inequality $K! \geq (K/e)^k$.) To handle (ii), we observe that from Lemma 3.3 and partial summation, the sum of the reciprocals of all $p$ satisfying (3.5) converges. Let $S$ denote this sum. Then the number of exceptions to (ii) is, for large $x$,

$$\leq \frac{x}{K!} \left( \sum_{\substack{p \leq x \\ p \text{ satisfies (3.5)}}} \frac{1}{p} \right)^K \leq x(eS/K)^K < x/(\log x)^A.$$

Hence, we can indeed assume (i) and (ii).

From (3.4), it now follows that there are at least $Z' - 2\frac{Z'}{4} = \frac{Z'}{4}$ primes $p$ for which $p \parallel n$ and for which $\Omega(p+1) > 16A \log A$. Hence,

$$\Omega(m) = \Omega(\sigma(n)) \geq \sum_{p \parallel n} \Omega(p+1) > 16A \log A \cdot \frac{Z'}{4} = 2A \log \log x.$$

But by Lemma 3.2, the number of $m \leq x$ with $\Omega(m)$ this large is $\ll_A x/(\log x)^A$. This completes the proof of Theorem 1.3 for large $x$. If $x$ is bounded in terms of $A$, then the theorem is trivial. $\qquad\square$

**Remark.** The method given here can be pushed to yield the more explicit result that the count of $m \leq x$ that occur as additive endpoints is smaller than

$$x/\exp\left( c_8 \log \log x \frac{\log \log \log x}{\log \log \log \log x} \right).$$

3.2. **The lower bound in Theorem 1.3.** The lower bound in Theorem 1.3 will be deduced from the following proposition, which may be of interest outside of this context.

**Proposition 3.4.** *Let $A > 0$. There is a constant $c = c(A)$ so that the following holds. If $x$ is sufficiently large, say $x > x_0(A, c)$, then any subset $\mathscr{S} \subset [1, x]$ with*

$$\#\mathscr{S} \leq x/\exp(c(\log \log x)^3)$$

*satisfies*

$$\#\sigma^{-1}(\mathscr{S}) \leq x/(\log x)^A.$$

*Here $\sigma^{-1}(\mathscr{S})$ denotes the set of $n$ with $\sigma(n) \in \mathscr{S}$.*

**Remark.** It is perhaps surprising that one cannot improve the upper bound on $\#\sigma^{-1}(\mathscr{S})$ very much, even if one assumes that $\mathscr{S}$ consists of only a single element! Indeed, plausible conjectures about the distribution of smooth shifted primes $p + 1$ (such as what would follow from the Elliott–Halberstam conjecture) imply that for all large $x$, there is a singleton set $\mathscr{S} \subset [1, x]$ with $\#\sigma^{-1}(\mathscr{S}) > x^{1-\epsilon}$. (Here $\epsilon > 0$ is arbitrary but fixed.) For the Euler $\varphi$-function, this result is due to Erdős [1] (see also the exposition of Pomerance [10]); the $\sigma$-version can be proved similarly, replacing $p - 1$ with $p + 1$ when necessary.

To apply Proposition 3.4 to the case of the practical numbers, it is convenient to recall Gronwall's determination of the maximal order of the sum-of-divisors function $\sigma$ [5, Theorem 323, p. 350].

**Lemma 3.5.** *We have* $\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^\gamma$.

*Proof of the lower bound in Theorem 1.3.* Let $x$ be large. By Lemma 3.5, if $n \le \frac{x}{2\log\log x}$, then $\sigma(n) \le x$. (We use here that $e^\gamma < 2$.) Thus, with $\mathscr{S}$ the set of additive endpoints not exceeding $x$,

$$\#\sigma^{-1}(\mathscr{S}) \ge PR\left(\frac{x}{2\log\log x}\right) \gg \frac{x}{(\log x)(\log\log x)},$$

using the lower estimate in (1.3) for the last step. The desired lower bound on $\#\mathscr{S}$ now follows from (the contrapositive of) Proposition 3.4, with $A = 1.1$. $\square$

The rest of this section is devoted to the proof of Proposition 3.4. The proof rests on a $\sigma$-analogue of a result for the Euler function appearing in a paper of Luca and the first author [7, Lemma 2.1].

**Lemma 3.6.** *Let $x \ge 3$. Let $d$ be a squarefree natural number with $d \le x$. The number of $n$ for which $d \mid \sigma(n)$ and $\sigma(n) \le x$ is*

$$\le \frac{x}{d}(c_9 \log x)^{3\omega(d)}.$$

*Proof.* If $d = 1$, the result is clear. Suppose that $d > 1$. Let $n$ be an integer for which $\sigma(n) \in [1, x]$ is a multiple of $d$. Write the prime factorization of $n$ in the form $n = \prod_i p_i^{e_i}$. Since $d \mid \sigma(n)$, there is a factorization $d = d_1 d_2 \cdots$ for which each $d_i \mid \sigma(p_i^{e_i})$. Discarding those terms with $d_i = 1$ and relabeling, we can assume that $d = d_1 \cdots d_\ell$, where each $d_i > 1$. Clearly, $\ell \le \omega(d)$.

We now fix the factorization $d = d_1 \cdots d_\ell$ and count the number of corresponding $n$. This count does not exceed

$$(3.6) \qquad x \prod_{i=1}^{\ell} \left( \sum_{\substack{p^e:\ \sigma(p^e)\le x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \right).$$

We proceed to estimate the inner sum in (3.6). If $d_i \mid \sigma(p^e)$, then $\sigma(p^e) = d_i m$, with $m \le x/d_i$. Since $\sigma(p^e) = 1 + p + \cdots + p^e \le 2p^e$,

$$\sum_{\substack{p^e:\ \sigma(p^e)\le x \\ d_i \mid \sigma(p^e)}} \frac{1}{p^e} \le \frac{2}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \sum_{p^e:\ \sigma(p^e)=md_i} 1.$$

For each fixed $e \ge 1$, there is at most one prime $p$ with $\sigma(p^e) = md_i$; moreover, since $md_i \le x$, there are no such $p$ once $e > \log x/\log 2$. Thus,

$$\frac{2}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \sum_{p^e:\ \sigma(p^e)=md_i} 1 \ll \frac{\log x}{d_i} \sum_{m \le x/d_i} \frac{1}{m} \ll \frac{(\log x)^2}{d_i}.$$

Inserted back into (3.6), we find that for a certain absolute constant $C > 1$, the number of $n$ corresponding to the given factorization is at most

$$x \prod_{i=1}^{\ell} \frac{C(\log x)^2}{d_i} = \frac{x}{d} C^\ell (\log x)^{2\ell} \le \frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)}.$$

Finally, we sum over unordered factorizations of $d$ into parts $> 1$. Since $d$ is squarefree, the number of such factorizations is precisely $B_{\omega(d)}$, where $B_k$ denotes the $k$th Bell number

(the number of set partitions of a $k$-element set). Thinking combinatorially, we have the crude bound $B_k \leq k^k$, and so the total number of $n$ which arise is

$$\leq \omega(d)^{\omega(d)} \left( \frac{x}{d} C^{\omega(d)} (\log x)^{2\omega(d)} \right) = \frac{x}{d} (C\omega(d)(\log x)^2)^{\omega(d)}.$$

By definition, we have $\omega(d) \leq \Omega(d) \leq \log x / \log 2$, where the final inequality follows from the simple observation that $2^{\Omega(d)} \leq d \leq x$. This proves our lemma with $c_9 = (C/\log 2)^{1/3}$. $\qquad \square$

**Lemma 3.7.** *Fix $A \geq 3$. The number of $n \leq x$ for which*

$$(3.7) \qquad\qquad \Omega(\sigma(n)) \geq 8A^2 (\log\log x)^2$$

*is $o(x/(\log x)^A)$, as $x \to \infty$.*

*Proof.* We may suppose that $\omega(n) \leq 2A \log\log x$. Indeed, Lemma 3.2 shows that for $x \geq 3$, the number of $n \leq x$ not satisfying the stronger inequality $\Omega(n) \leq 2A \log\log x$ is

$$\ll \frac{A \log\log x}{2^{A \log\log x}} x \log x \ll_A \frac{x \log\log x}{(\log x)^{2A \log 2 - 1}}.$$

Since $A \geq 3$, the exponent $2A \log 2 - 1 > A$, and so this upper bound is $o(x/(\log x)^A)$.

Writing $\Omega(\sigma(n)) = \sum_{p^e \| n} \Omega(\sigma(p^e))$, we thus deduce that if (3.7) holds, then

$$(3.8) \qquad\qquad \Omega(\sigma(p^e)) \geq \frac{8A^2 (\log\log x)^2}{2A \log\log x} = 4A \log\log x$$

for some prime power $p^e \| n$.

Suppose first that $e > 1$. Then (for large $x$) the squarefull part of $n$ is of size at least

$$(3.9) \qquad\qquad p^e \geq \frac{1}{2} \sigma(p^e) \geq \frac{1}{2} 2^{\Omega(\sigma(p^e))} \geq \frac{1}{2} 2^{4A \log\log x} > (\log x)^{5A/2}.$$

But then the number of possibilities for $n \leq x$ is $\ll x/(\log x)^{5A/4}$, and so in particular is $o(x/(\log x)^A)$. On the other hand, if $e = 1$, then (3.8) implies that $n$ is divisible by some prime $p$ with $\Omega(p+1) \geq 4A \log\log x$. For each such $p$, the number of corresponding $n$ is $\leq x/p < 2x/(p+1)$. Summing over $p$, we find that the total number of such $n \leq x$ is at most

$$2x \sum_{\substack{d \leq x \\ \Omega(d) \geq 4A \log\log x}} \frac{1}{d}.$$

Put $Z := 4A \log\log x$; by partial summation, along with Lemma 3.2 and the final inequality in (3.9), this upper bound is

$$\ll x \frac{Z}{2^Z} \int_2^x \frac{\log t}{t} \, dt \ll x (\log x)^2 \frac{Z}{2^Z} \ll_A x \frac{(\log x)^2 \log\log x}{(\log x)^{5A/2}},$$

and so is $o(x/(\log x)^A)$, as $x \to \infty$. This completes the proof. $\qquad \square$

**Lemma 3.8.** *Let $x \geq 3$, and let $z \geq 1$. The number of $n \leq x$ with $\sigma(n)$ divisible by $p^2$ for some prime $p > z$ is $\ll x(\log x)^2 z^{-1/2}$.*

*Proof.* If $p^2 \mid \sigma(n)$, then either $p \mid \sigma(q^e)$ for a proper prime power $q^e$ exactly dividing $n$, or there are two distinct primes $q_1$ and $q_2$ exactly dividing $n$ with $q_1, q_2 \equiv -1 \pmod{p}$. In the former case, $n$ has a squarefull divisor of size $\geq q^e \geq \frac{1}{2}\sigma(q^e) > p/2 > z/2$. The

number of such $n$ is $\ll xz^{-1/2}$, which is acceptable for us. For a given $p$, the number of $n$ arising in the second case is

$$\leq x\left(\sum_{\substack{q\leq x \\ q\equiv -1 \pmod p}} \frac{1}{q}\right)^2 \leq x\left(\sum_{j\leq x/p}\frac{1}{pj-1}\right)^2 \ll x(\log x)^2 p^{-2}.$$

Summing over $p > z$, we find that the total number of $n$ that can arise from this case is $\ll x(\log x)^2 z^{-1}$, which is also acceptable. $\qquad\square$

*Proof or Proposition 3.4.* We may suppose that our fixed constant $A$ satisfies $A \geq 5$. We will show that for such $A$, the proposition holds with $c(A) = 50A^3$.

Let $\mathscr{S}_1$ consist of those $m \in \mathscr{S}$ for which either

(i) $m \leq x/(\log x)^{2A}$, or
(ii) $\Omega(m) \geq 8A^2(\log\log x)^2$, or
(iii) $p^2 \mid m$ for some $p > (\log x)^{3A}$.

We let $\mathscr{S}_2$ consist of the remaining elements of $\mathscr{S}$. By Lemmas 3.7 and 3.8, the size of $\sigma^{-1}(\mathscr{S}_1)$ is $o(x/(\log x)^A)$ as $x \to \infty$ (uniformly in the choice of $\mathscr{S}$).

We turn now to $\mathscr{S}_2$. To each $m \in \mathscr{S}_2$, we associate the divisor $m'$ of $m$ defined by

$$m' := \prod_{\substack{p^e \| m \\ p > (\log x)^{3A}}} p.$$

Then $m'$ is squarefree, and

(3.10) $$\omega(m') \leq \Omega(m) < 8A^2(\log\log x)^2.$$

Moreover, assuming that $x$ is large, since $m > (x/\log x)^{2A}$,

(3.11) $$m' \geq m/((\log x)^{3A})^{\Omega(m)} > \frac{x/(\log x)^{2A}}{\exp(24A^3(\log\log x)^3)} > x/\exp(25A^3(\log\log x)^3).$$

We bound the number of $\sigma$-preimages of $m$ from above by the number of $n$ for which $\sigma(n) \in [1, x]$ is a multiple of $m'$. By Lemma 3.6, along with (3.10) and (3.11), the number of such $n$ is

$$\leq \frac{x}{m'}(c_9\log x)^{3\omega(m')} \leq \exp(25A^3(\log\log x)^3)(c_9\log x)^{24A^2(\log\log x)^2}$$

$$\leq \exp(49A^3(\log\log x)^3),$$

say. Summing over the elements of $\mathscr{S}_2$, we find that

$$\#\sigma^{-1}(\mathscr{S}_2) \leq \exp(49A^3(\log\log x)^3) \cdot \#\mathscr{S}_2.$$

So if we assume that $\#\mathscr{S} \leq x/\exp(50A^3(\log\log x)^3)$, then $\#\sigma^{-1}(\mathscr{S}_2) = o(x/(\log x)^A)$, as $x \to \infty$. Combined with our earlier estimate on the size of $\sigma^{-1}(\mathscr{S}_1)$, this shows that $\#\sigma^{-1}(\mathscr{S}) \leq x/(\log x)^A$ once $x$ is sufficiently large. $\qquad\square$

## 4. PROOF OF THEOREM 1.5

The key to the proof of Theorem 1.5 is the following inequality of Robin [12, Théorème 2].

**Lemma 4.1.** *For each natural number $n \geq 3$,*

$$\sigma(n) \leq e^\gamma n \log\log n + 0.6483\frac{n}{\log\log n}.$$

*Proof of Theorem 1.5.* Suppose for the sake of contradiction that $f(n) \geq H(n)$ but that $n$ is not practical. We assume to begin with that $n > 14$, treating small $n$ at the end of the proof. Let $d$ be the practical component of $n$, and write $n = dq$. Then $q > 1$, and

$$P^-(q) > \sigma(d) + 1 = f(n) + 1 > H(n) > n^{1/2},$$

where in fact the last inequality holds for all $n > 6$. It follows that $q$ is prime and $P^-(q) = q$. Hence, $H(n) < q = n/d$, and so

$$d < \frac{n}{H(n)}.$$

Also, since $\sigma(d) = f(n) \geq H(n)$, we have

$$q = n/d \leq \frac{n}{d} \frac{\sigma(d)}{H(n)}.$$

Multiplying the last two displayed inequalities shows that

$$n = dq \leq \frac{\sigma(d)}{d} \left( \frac{n}{H(n)} \right)^2 = \frac{\sigma(d)}{d} n (e^\gamma \log \log n)^{-1},$$

and so

(4.1) $$\frac{\sigma(d)}{d} \geq e^\gamma \log \log n.$$

Since $q > n^{1/2}$ and $n = dq$, we have that $q > d$, and so

$$\log \log n = \log \log (qd) > \log \log (d^2) = \log \log d + \log 2;$$

thus, (4.1) gives

$$\frac{\sigma(d)}{d} \geq e^\gamma \log \log d + e^\gamma \log 2$$

(4.2) $$> e^\gamma \log \log d + 1.2345.$$

We now derive a contradiction to Robin's inequality. We can assume that $d \geq 6$; otherwise, $\sigma(d)/d \leq 7/4$, and (4.1) then implies that $n \leq 14$, contrary to hypothesis. By Lemma 4.1,

$$\frac{\sigma(d)}{d} \leq e^\gamma \log \log d + \frac{0.6483}{\log \log d}.$$

Combining this inequality with (4.2), we obtain $0.6483/ \log \log d > 1.2345$. But this fails for all $d \geq 6$. This contradiction completes the proof for $n > 14$.

It remains to treat the cases when $3 < n \leq 14$. For odd $n > 3$, the hypotheses of the theorem are never satisfied, since $f(n) = 1 < H(5) \leq H(n)$. So the only possible exceptions to the theorem have $n$ even. The non-practical even values of $n \leq 14$ are $n = 10$ and $n = 14$, and in both cases, $f(n) = 3 < H(n)$, so the theorem holds. $\square$

## References

[1] P. Erdős, *On the normal number of prime factors of $p-1$ and some related problems concerning Euler's $\varphi$-function*, Quart. J. Math. **6** (1935), 205–213.

[2] _____, *On a Diophantine equation*, Mat. Lapok **1** (1950), 192–210.

[3] H. Halberstam and K. F. Roth, *Sequences*, second ed., Springer-Verlag, New York, 1983.

[4] R. R. Hall and G. Tenenbaum, *Divisors*, Cambridge Tracts in Mathematics, vol. 90, Cambridge University Press, Cambridge, 1988.

[5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008.

[6] M. Hausman and H. N. Shapiro, *On practical numbers*, Comm. Pure Appl. Math. **37** (1984), no. 5, 705–713.

[7] F. Luca and P. Pollack, *An arithmetic function arising from Carmichael's conjecture*, J. Théorie Nombres Bordeaux **23** (2011), 697–714.

[8] F. Luca and C. Pomerance, *Irreducible radical extensions and Euler-function chains*, Combinatorial number theory, de Gruyter, Berlin, 2007, pp. 351–361.

[9] M. Margenstern, *Les nombres pratiques: théorie, observations et conjectures*, J. Number Theory **37** (1991), no. 1, 1–36.

[10] C. Pomerance, *Two methods in elementary analytic number theory*, Number theory and applications (Banff, AB, 1988), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 265, Kluwer Acad. Publ., Dordrecht, 1989, pp. 135–161.

[11] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin, 1957.

[12] G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. (9) **63** (1984), no. 2, 187–213.

[13] E. Saias, *Entiers à diviseurs denses. I*, J. Number Theory **62** (1997), no. 1, 163–191.

[14] W. Sierpiński, *Sur une propriété des nombres naturels*, Ann. Mat. Pura Appl. (4) **39** (1955), 69–74.

[15] A. K. Srinivasan, *Practical numbers*, Current Science **17** (1948), 179–180.

[16] B. M. Stewart, *Sums of distinct divisors*, Amer. J. Math. **76** (1954), 779–785.

[17] G. Tenenbaum, *Sur un problème de crible et ses applications*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 1–30.

[18] _____, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics, vol. 46, Cambridge University Press, Cambridge, 1995.

[19] _____, *Sur un problème de crible et ses applications. II. Corrigendum et étude du graphe divisoriel*, Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 2, 115–127.

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