Divisor-sum fibers

Lola Thompson

Oberlin College

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The $s(n)$ function

Definition

Let $s(n)$ denote the sum of proper divisors of n.

Example: $s(p) = 1$ for any prime p

Example: $s(12) = 1 + 2 + 3 + 4 + 6 = 16$

We can write $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum-of-divisors function.

Perfect numbers

Pythagoras observed:

$$s(\mathbf{6}) = 1 + 2 + 3 = \mathbf{6}.$$

Definition

n is **perfect** if $s(n) = n$.

Amicable pairs

Definition

If $s(\mathbf{n}) = \mathbf{m}$, $s(\mathbf{m}) = \mathbf{n}$, and $\mathbf{m} \neq \mathbf{n}$, then $\mathbf{n}$ and $\mathbf{m}$ form an **amicable pair**.

Example (Pythagoras):

$$s(\mathbf{220}) = \mathbf{284}, \quad s(\mathbf{284}) = \mathbf{220}.$$

Iterates of s

We can view s as a dynamical system, looking at its iterates:

$6 \to 6$

$10 \to 8 \to 7 \to 1$

$12 \to 16 \to 15 \to 9 \to 4 \to 3 \to 1$

$28 \to 28$

$220 \to 284 \to 220$

$276 \to 396 \to \cdots$

Pythagoras studied 1-cycles (perfect numbers) and 2-cycles (amicable pairs).

Results on the iterates of s

Theorem (Hornfeck & Wirsing, 1957)

The number of $n \leq x$ with $s(n) = n$ is at most x^ϵ.

Theorem (Pomerance, 2014)

The number of $n \leq x$ with n in a 2-cycle is at most $x/\exp((\log x)^{1/2})$ for x large.

Motivating questions

"Studying the comparison of $s(n)$ to n led to theorems of Schoenberg, Davenport, and Erdős-Wintner, and the birth of probabilistic number theory." -Carl Pomerance

In this talk, we will focus on two particular questions concerning the function $s(n)$:

1. Which numbers are of the form $s(n)$?
2. How large is the set $s^{-1}(n)$?

The image of s

Erdős was the first to consider questions about the image of s.

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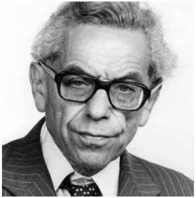
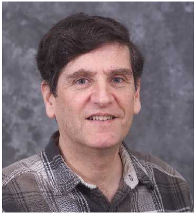
So almost all odd numbers ≥ 9 are values of s.

What about even numbers?


Theorem (Erdős, 1973)

A positive proportion of even integers are missing from the image of s.

Theorem (Luca & Pomerance, 2014)

A positive proportion of even integers are in the image of s.

The image of s

The function s can map sets of asymptotic density 0 to sets with positive asymptotic density.

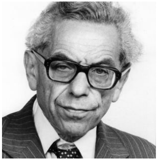
Example If $\mathcal{A} = \{pq : p, q \text{ prime}\}$ then $\mathcal{A}$ has asymptotic density 0 but $s(\mathcal{A})$ has asymptotic density $1/2$.

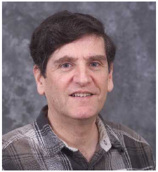
Example Erdős constructed sets $\mathcal{A}$ of positive density such that $s^{-1}(\mathcal{A})$ not only has density 0 but is, in fact, empty.

The preimage of s

What can be said about $s^{-1}(\mathcal{A})$ when $\mathcal{A}$ has asymptotic density 0?

The EGPS Conjecture






Conjecture (Erdős, Granville, Pomerance, Spiro, 1990)

Let $\mathcal{A}$ be a set with asymptotic density 0. Then $s^{-1}(\mathcal{A})$ also has asymptotic density 0.

Consequences of EGPS

Some consequences of EGPS (if true):

1. For each fixed positive integer k, but for a set of n with density 0, if $s(n) < n$ then $s_k(n) < s_{k-1}(n) < \cdots < n$ where s_j is the j-fold iterate of s.

2. For each integer $k \geq 2$, there is a set $\mathcal{A}_k$ of asymptotic density 1 such that

$$\frac{1}{x} \sum_{\substack{n \leq x \\ n \in \mathcal{A}_k}} \log(s_k(n)/s_{k-1}(n)) \to \beta,$$

as $x \to \infty$, where β comes from a theorem of Bosma and Kane: $\frac{1}{x} \sum_{n \leq x} \log(s(2n)/2n) \sim \beta$ as $x \to \infty$.

Special cases of EGPS

Some special cases of EGPS have been proven:

- (Pollack, 2014) If $\mathcal{A}$ is the set of primes, then

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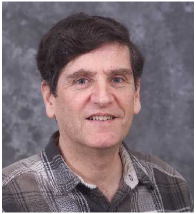
If $\mathcal{A}_\epsilon = \{m : |\omega(m) - \log\log m| > \epsilon \log\log m\}$ then $s^{-1}(\mathcal{A}_\epsilon)$ has density 0.
- (Pollack, 2015) If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has density 0.

Partial Progress on EGPS


Theorem (Pollack, Pomerance, T., 2017)

Let $\epsilon \to 0$ *as* $x \to \infty$. *Suppose* $\mathcal{A}$ *is a set of at most* $x^{1/2+\epsilon}$ *positive integers. Then, as* $x \to \infty$,

$$\#\{n \le x : s(n) \in \mathcal{A}\} = o_\epsilon(x)$$

uniformly in $\mathcal{A}$.

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- If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has density 0.
- If $\mathcal{A}$ is the set of squares, then $s^{-1}(\mathcal{A})$ has density 0.

Disproving a stronger EGPS conjecture

EGPS point out that their conjecture would be a consequence of the following assertion about the sizes of elements in a fiber.

Hypothesis (Erdős, Granville, Pomerance, Spiro, 1990)

For each positive number θ there exists a constant C_θ such that for all positive integers m there exist at most C_θ numbers $n \leq \theta m$ with $s(n) = m$.

Proof Sketch

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Note: $m = s(2pq) \geq pq$, so that each preimage $2pq \leq 2m$.

Thus, the Strong EGPS conjecture fails for $\theta = 2$.

Disproving a stronger EGPS conjecture

Using an elaboration on these methods, we show:

Theorem (Pollack, Pomerance, T., 2017)

There is a constant $c > 0$ for which the following holds. Let α and ϵ be positive real numbers. There are infinitely many m with at least $\exp(c \log m / \log\log m)$ s-preimages that lie in the interval $(\alpha(1-\epsilon)m, \alpha(1+\epsilon)m)$.

Key Lemma

We use the following generalization of Erdős-Prachar:

Lemma (Pollack, Pomerance, T., 2017)

There is a positive absolute constant c such that, for all $a, b \in \mathbb{Z}$ with $a \neq 0$ and $b > 0$, there are infinitely many integers k with more than $\exp(c \log k / \log \log k)$ representations as $(bp + a)(bq + a)$ with p, q primes.

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Let $0 < \epsilon < 1$. It is well-known that the values of $s(n)/n$ are dense in $(0, \infty)$.

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Write $n = n_0 pq$, where p, q are distinct primes not dividing n_0. Then

$$\begin{aligned} s(n_0 pq) &= \sigma(n_0)(p+1)(q+1) - n_0 pq \\ &= s(n_0)pq + \sigma(n_0)(p+q+1), \end{aligned}$$

so that

$$\begin{aligned} s(n_0)s(n_0 pq) = (s(n_0)p + \sigma(n_0))(s(n_0)q + \sigma(n_0)) \\ + s(n_0)\sigma(n_0) - \sigma(n_0)^2. \end{aligned}$$

Proof Sketch

By Key Lemma, there are infinitely many $k \in \mathbb{Z}$ with more than $\exp(c \log k / \log\log k)$ representations of the form

$$k = (s(n_0)p + \sigma(n_0))(s(n_0)q + \sigma(n_0))$$

with p, q distinct.

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Thus, m has at least $\exp(c \log m / \log \log m)$ preimages $n = n_0pq$ in $((1 - \epsilon)\alpha m, (1 + \epsilon)\alpha m)$.

Thank you!