Bounded gaps between primes and the length spectra of arithmetic hyperbolic surfaces

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July 27, 2017

Collaborators

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Geometric Motivation

Let M be a compact Riemannian manifold.

The **Laplace eigenvalue spectrum** of M, denoted $\mathcal{E}(M)$, is the multiset of eigenvalues of the Laplacian of M.

The **geodesic length spectrum**, denoted $LS(M)$, is the multiset of lengths of closed geodesics on M with fundamental group $\pi_1(M)$.

Inverse spectral geometry asks for the extent to which the spectra of M determine its geometry and topology.

Geometric terminology

Definition

Two manifolds are **commensurable** if and only if they have a common finite degree covering space.

Definition

Two manifolds are **isometric** if there is an isometry between them.

Definition

M and N are **isospectral** if $\mathcal{E}(M) = \mathcal{E}(N)$, and **length isospectral** if $LS(M) = LS(N)$.

Inverse Problems

Some natural inverse problems:

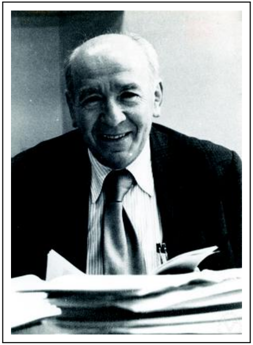
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, is M isometric to N?
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, are M and N commensurable?
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, what can be said about N?

We can ask the same questions for $\mathcal{E}(M)$.

Can you hear the shape of a drum?

Leon Green (1960) asked if the spectrum of M determines its isometry class.

The spectrum of M is essentially the collection of frequencies produced by a drumhead shaped like M.



Mark Kac (1966) popularized this question for planar domains:

Can you hear the shape of a drum?

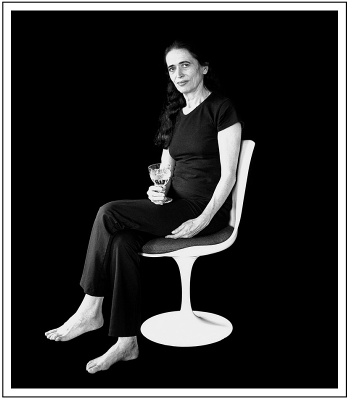
Can you hear the shape of a drum?

Milnor had already shown that isometry class is not, in general, a spectral invariant.

Theorem (Milnor, 1964)

There exist lattices $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{16}$ *such that the tori* $\mathbb{R}^{16}/\Gamma_1$ *and* $\mathbb{R}^{16}/\Gamma_2$ *are isospectral but not isometric.*

Known Results



Theorem (Vigneras, 1980)

There exist isospectral but not isometric hyperbolic 2 and 3-manifolds.

Can you hear the shape of a drum?

Kac's question was finally answered in 1992.

Theorem (Gordon, Webb, Wolpert, 1992)

One cannot hear the shape of a drum.

Geometric motivation

Our work

Where number theory enters the picture

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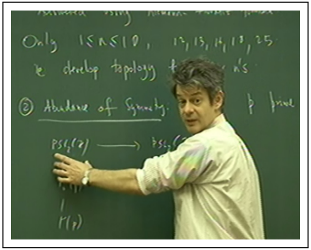
Known Results (for surfaces)



Theorem (Reid, 1992)

If M is an arithmetic, hyperbolic surface and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

Lubotzky, Samuels and Vishne (2005) showed: if one considers the symmetric space of $\mathrm{PGL}_n(\mathbb{R})$ or $\mathrm{PGL}_n(\mathbb{C})$ then, for $n > 3$, Reid's result is false and one can obtain arbitrarily large families of isospectral yet non-commensurable arithmetic manifolds.

Known Results (for 3-manifolds)

In the context of hyperbolic 3-manifolds, Reid's result is due to Chinburg, Hamilton, Long and Reid.

Theorem (Chinburg, Hamilton, Long, Reid, 2008)

If M and N are arithmetic hyperbolic 3-manifolds and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

The length spectrum of hyperbolic 3-manifolds

Theorem (Futer and Millichap, 2016)

For every sufficiently large $n > 0$ there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds $\{N_n, N_n^\mu\}$ such that:

1. $\mathrm{vol}(N_n) = \mathrm{vol}(N_n^\mu)$, *where this volume grows coarsely linearly with n.*
2. *The (complex) length spectra of N_n and N_n^μ agree up to length n.*
3. *N_n and N_n^μ have at least e^n/n closed geodesics up to length n.*
4. *N_n and N_n^μ are not commensurable.*

Our work

Motivating Question: If the length spectra have a great deal of overlap, must the corresponding manifolds be commensurable?

Bounded gaps and length spectra

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Our work

Let $\pi(V,S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 3–orbifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2014)

If $\pi(V,S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V,S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Our new result

Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Let M be an arithmetic hyperbolic 3–orbifold which is derived from a quaternion algebra and let S be a finite subset of the length spectrum of M. Suppose that $\pi(V, S) \to \infty$ as $V \to \infty$. Then, for every $k \geq 2$, there is a constant $C > 0$ such that there are infinitely many k–tuples $M_1, \ldots, M_k$ of arithmetic hyperbolic 3–orbifolds which are pairwise non-commensurable, have length spectra containing S, and volumes satisfying $|\mathrm{vol}(M_i) - \mathrm{vol}(M_j)| < C$ for all $1 \leq i, j \leq k$.

Where number theory enters the picture

Let M be an arithmetic hyperbolic 3–orbifold arising from (K, A) with fundamental group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$.

The closed geodesics

$$c_\gamma \colon S^1 \longrightarrow M$$

on M are in bijection with the Γ–conjugacy classes $[\gamma]_\Gamma$ of hyperbolic elements γ in Γ.

The associated geodesic length $\ell(c_\gamma)$ is given by

$$\cosh \frac{\ell(c_\gamma)}{2} = \pm \frac{\mathrm{Tr}(\gamma)}{2}.$$

Where number theory enters the picture

We denote by λ_γ the unique eigenvalue of γ with $|\lambda_\gamma| > 1$.

Each closed geodesic c_γ determines a maximal subfield K_γ of the quaternion algebra A.

Specifically, $K_\gamma = K(\lambda_\gamma)$.

As Γ is arithmetic, λ_γ is in $\mathcal{O}^1_{K_\gamma}$.

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Borel's Covolume Formula:

$$\mathrm{vol}(\mathbf{H}^3/\Gamma_{\mathcal{O}_K^1}) = \frac{|\Delta_K|^{3/2}\zeta_K(2)}{(4\pi^2)^{n_K-1}} \prod_{P\in\mathrm{Ram}_f(A)} (N(P)-1)\,.$$

BCF shows: If two orbifolds have the same field of definition K but their associated quaternion algebras ramify at different primes, then their volumes will differ by some function of the norm of the primes that ramify.

So, primes with bounded gaps between them produce orbifolds with volumes lying in bounded length intervals.

Where number theory enters the picture

We also want these orbifolds to have length spectra containing S, which will happen if and only if the quadratic extensions K_γ embed into the quaternion algebras.

One can arrange this by choosing primes (ramifying in the quaternion algebras) to lie in certain appropriately-chosen Chebotarev sets.

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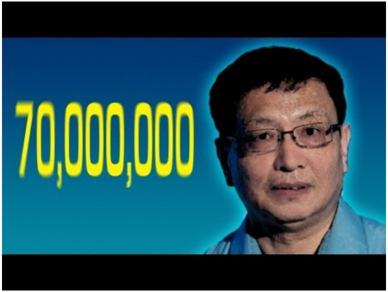
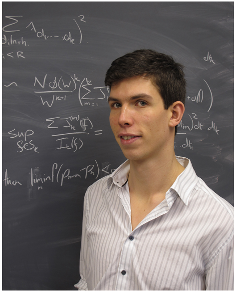
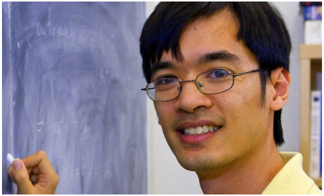


Theorem (Zhang, 2013)

There are infinitely many pairs of primes that differ by at most $70,000,000$.

Bounded gaps between primes

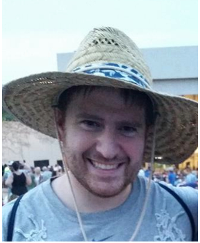




Theorem (Maynard-Tao, 2013)

Let $m \geq 2$. There for any admissible k-tuple $\mathcal{H} = (h_1, ..., h_k)$ with "large enough" k (relative to m), there are infinitely many n such that at least m of $n + h_1, ..., n + h_k$ are prime.

Bounded gaps between primes in Chebotarev sets



Theorem (Thorner, 2014)

Let $K/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and discriminant Δ, and let $\mathcal{C}$ be a conjugacy class of G. Let $\mathcal{P}$ be the set of primes $p \nmid \Delta$ for which $\left(\frac{K/\mathbb{Q}}{p}\right) = \mathcal{C}$. Then there are infinitely many pairs of distinct primes $p_1, p_2 \in \mathcal{P}$ such that $|p_1 - p_2| \leq c$, where c is a constant depending on $G, \mathcal{C}, \Delta$.

Bounded gaps between primes in Chebotarev sets

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Theorem (Linowitz, McReynolds, Pollack, T., 2017)

Let L/K be a Galois extension of number fields, let $\mathcal{C}$ be a conjugacy class of $\mathrm{Gal}(L/K)$, and let k be a positive integer. Then, for a certain constant $c = c_{L/K,\mathcal{C},k}$, there are infinitely many k–tuples $P_1, \dots, P_k$ of prime ideals of K for which the following hold:

1. $\left(\frac{L/K}{P_1}\right) = \cdots = \left(\frac{L/K}{P_k}\right) = \mathcal{C}$,
2. *$P_1, \dots, P_k$ lie above distinct rational primes,*
3. *each of $P_1, \dots, P_k$ has degree 1,*
4. *$|N(P_i) - N(P_j)| \leq c$, for each pair of $i, j \in \{1, 2, \dots, k\}$.*

This extends Thorner's work, which implies the case where $K = \mathbb{Q}$.

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Thank you!