VARIATIONS ON A QUESTION CONCERNING THE DEGREES OF DIVISORS OF $x^n - 1$

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ABSTRACT. In this paper, we examine a natural question concerning the divisors of the polynomial $x^n - 1$: "How often does $x^n - 1$ have a divisor of every degree between 1 and n?" In a previous paper, we considered the situation when $x^n - 1$ is factored in $\mathbb{Z}[x]$. In this paper, we replace $\mathbb{Z}[x]$ with $\mathbb{F}_p[x]$, where p is an arbitrary-but-fixed prime. We also consider those n where this condition holds for all p.

1. INTRODUCTION AND STATEMENT OF RESULTS

Which polynomials have divisors of every degree in a given polynomial ring? In a previous paper [8], we answered this question in $\mathbb{Z}[x]$ for the family of polynomials $f(x) = x^n - 1$, where n ranges over all positive integers. We defined an integer n to be *φ-practical* if the polynomial $x^n - 1$ has a divisor in $\mathbb{Z}[x]$ of every degree up to n and we showed that, if $F(X) = \#\{n \leq X : n \text{ is } \varphi\text{-practical}\}$, then there exist two positive constants c_1 and c_2 such that

$$c_1 \frac{X}{\log X} \leq F(X) \leq c_2 \frac{X}{\log X}. \tag{1.1}$$

In this paper, we will examine the factorization of $x^n - 1$ over other rings. For each rational prime p, we will define an integer n to be *p-practical* if $x^n - 1$ has a divisor in $\mathbb{F}_p[x]$ of every degree less than or equal to n. In order to better understand the relationship between φ-practical and p-practical numbers, we will define an intermediate set of numbers which we shall call the λ-practical numbers. An integer n is *λ-practical* if and only if it is p-practical for every rational prime p. Clearly each φ-practical number is λ-practical. In Sections 2 and 3, we will give alternative characterizations of the p-practical and λ-practical numbers that are often easier to work with.

The main goal of this paper is to examine the relative sizes of the sets of φ-practical, λ-practical and p-practical numbers. Accordingly, the remaining sections take the following form. In Section 4, we will develop some theory on the structure of λ-practical numbers and show that there are infinitely many λ-practical numbers that are not φ-practical. We will go one step further in Section 5 and prove:

Theorem 1.1. *For X sufficiently large, the order of magnitude of λ-practicals in $[1, X]$ that are not φ-practical is $\frac{X}{\log X}$.*

In Section 6, we examine the relationship between p-practicals and λ-practicals, culminating in a proof of the following theorem:

Theorem 1.2. *For every rational prime p and for X sufficiently large, the order of magnitude of p-practicals in $[1, X]$ that are not λ-practical is at least $\frac{X}{\log X}$.*

We remark that the order of magnitude described in Theorem 1.2 is deemed to be *at least* $\frac{X}{\log X}$ (as opposed to *precisely* $\frac{X}{\log X}$) because of the fact that we still do not know the true order of magnitude of the p-practical numbers. In another paper [9], we show (assuming the validity of the Generalized Riemann Hypothesis) that

$$\frac{X}{\log X} \ll \#\{n \leq X : n \text{ is } p\text{-practical}\} \ll X\sqrt{\frac{\log\log X}{\log X}}.$$

As a result, Theorem 1.2 tells us that the order of magnitude of p-practicals that are not λ-practical is between $\frac{X}{\log X}$ and $X\sqrt{\frac{\log\log X}{\log X}}$ (with the upper bound holding provided that the Generalized Riemann Hypothesis is valid).

Throughout this paper, we will make use of the following notation. Let n be a positive integer. Let p and q, as well as any subscripted variations, be primes. We will use $P(n)$ to denote the largest prime factor of n, with $P(1) = 1$. Moreover, we will use $P^-(n)$ to denote the smallest prime factor of n, with $P^-(1) = +\infty$. Let $\tau(n)$ denote the number of positive divisors of n, and let $\Omega(n)$ denote the number of prime factors of n which are not necessarily distinct.

2. Background and preliminary results

In [8], we gave the following alternative characterization for the φ-practical numbers, which we state here as a lemma.

Lemma 2.1. *An integer n is φ-practical if and only if every m with $1 \leq m \leq n$ can be written in the form*

$$m = \sum_{d \in \mathcal{D}} \varphi(d),$$

where $\mathcal{D}$ is a subset of divisors of n.

It is not difficult to see Lemma 2.1: since

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where $\Phi_d(x)$ is the d^{th} cyclotomic polynomial (which is irreducible in $\mathbb{Z}[x]$ with degree $\varphi(d)$), we see that divisors of $x^n - 1$ correspond to subsets of divisors of n.

The term "φ-practical" was chosen in recognition of the connection between this alternative characterization of φ-practical numbers and the definition of a practical number. A. K.

Srinivasan coined the term "practical number" in 1948, defining an integer n to be *practical* if every m with $1 \leq m \leq \sigma(n)$ can be written as a sum of distinct positive divisors of n; that is, $m = \sum_{d \in \mathcal{D}} d$, where $\mathcal{D}$ is a subset of divisors of n. Six years later, B. M. Stewart [5] gave a simple necessary-and-sufficient condition for integers to be practical:

Lemma 2.2 (Stewart)**.** *If M is a practical number and p is a prime with $(p, M) = 1$, then $M' = p^k M$ is practical (for $k \geq 1$) if and only if $p \leq \sigma(M) + 1$. Moreover, if M is practical, so is $M/P(M)$.*

Stewart's condition gave rise to a number of results concerning the practical numbers, most notably a series of improvements on upper and lower bounds for the size of the set of practical numbers up to X. The tightest bounds were given by E. Saias in [3], who showed that there exist two constants C_1 and C_2 such that

$$C_1 \frac{X}{\log X} \leq PR(X) \leq C_2 \frac{X}{\log X}, \tag{2.1}$$

where $PR(X) = \#\{n \leq X : n \text{ is practical}\}$.

Recall the upper and lower bounds for $F(X)$ given in (1.1), which mirror Saias' bounds for the practical numbers. One of our aims in this paper will be to obtain a similar upper bound for the λ-practical numbers. As in the case of the φ-practical numbers, we will find it helpful to have alternative characterizations of the λ-practical and p-practical numbers in terms of their divisors. Let $\ell_a(n)$ denote the multiplicative order of $a \pmod n$ for integers a with $(a, n) = 1$. If $(a, n) > 1$, let $n_{(a)}$ denote the largest divisor of n that is coprime to a, and let $\ell_a^*(n) = \ell_a(n_{(a)})$. In particular, if $(a, n) = 1$ then $\ell_a^*(n) = \ell_a(n)$.

Lemma 2.3. *An integer n is p-practical if and only if every m with $1 \leq m \leq n$ can be written as $m = \sum_{d|n} \ell_p^*(d) n_d$, where n_d is an integer with $0 \leq n_d \leq \frac{\varphi(d)}{\ell_p^*(d)}$.*

To see the relationship between the two characterizations of p-practical numbers, recall the following well-known proposition (cf. [1, pg. 489, ex.20]):

Proposition 2.4. *The following two cases completely characterize the factorization of $\Phi_d(x)$ over $\mathbb{F}_p$:*

1) If $(d, p) = 1$, then $\Phi_d(x)$ decomposes into a product of distinct irreducible polynomials of degree $\ell_p(d)$ in $\mathbb{F}_p[x]$.

2) If $d = mp^k$, $(m, p) = 1$, then $\Phi_d(x) = \Phi_m(x)^{\varphi(p^k)}$ over $\mathbb{F}_p$.

Thus, the correspondence between the definitions follows from the fact that each cyclotomic polynomial $\Phi_d(x)$ dividing $x^n - 1$ factors into $\varphi(d)/\ell_p^*(d)$ irreducible polynomials of degree $\ell_p^*(d)$ over $\mathbb{F}_p[x]$.

As we will discuss in the next section, the λ-practical numbers can be defined in a similar manner. However, it takes a bit more work to prove this.

3. An alternative characterization for the λ-practical numbers

Just as we showed that the φ-practical and p-practical numbers have alternative characterizations that resemble the definition of a practical number, we can also show that the λ-practical numbers have such a characterization. Let $\lambda(n)$ denote the universal exponent of the multiplicative group of integers modulo n. We will show that the following theorem gives a criterion for an integer n to be λ-practical that is equivalent to the definition that we gave in Section 1:

Theorem 3.1. *An integer n is λ-practical if and only if we can write every integer m with $1 \leq m \leq n$ in the form $m = \sum_{d|n} \lambda(d) m_d$, where m_d is an integer with $0 \leq m_d \leq \frac{\varphi(d)}{\lambda(d)}$.*

Before presenting the proof, however, we will pause to ponder a related question. We can think of the set of integers n that are "p-practical for all primes p" as the intersection between all of the sets of integers that are p-practical. In addition to describing the intersection of these sets, we can also describe their union.

Proposition 3.2. *For each prime p, let S_p be the set of p-practical numbers. Then*

$$\bigcup_{p \text{ prime}} S_p = \mathbb{Z}_+.$$

In fact, we can prove a stronger result: it turns out that each integer n is p-practical for infinitely many values of p. Namely, for a given n, Dirichlet's Theorem on Primes in Arithmetic Progressions [2, pp. 119] implies that there are infinitely many primes p for which $p \equiv 1 \pmod{n}$. In other words, $\ell_p(n) = 1$ for infinitely many primes p, so $x^n - 1$ splits completely into linear factors in $\mathbb{F}_p[x]$ for infinitely many primes p. This argument implies that each integer n is p-practical for a positive proportion of the p's. We could also observe that, if $p \equiv -1 \pmod{n}$, then $\ell_p(n) = 2$, hence all of the irreducible factors of $x^n - 1$ have degree at most 2. Dirichlet's Theorem also guarantees the existence of infinitely many such primes. We remark that there are integers n for which the set of primes $p \equiv \pm 1 \pmod{n}$ are the only primes for which $x^n - 1$ has a divisor of every degree ($n = 5$ is the smallest such integer).

We will now show that $\bigcup_{p \text{ prime}} S_p$ is precisely the set of integers satisfying the conditions given in Theorem 3.1.

Lemma 3.3. *For all positive integers n, there exists a prime p such that $\ell_p^*(d) = \lambda(d)$ for all $d \mid n$.*

Proof. First, we will consider the case where $n = q^e$, where q is an odd prime. Each divisor of n is of the form $d = q^f$, with $0 \leq f \leq e$. Since $(\mathbb{Z}/q^f\mathbb{Z})^\times$ is cyclic, there must be some element $a \in (\mathbb{Z}/q^e\mathbb{Z})^\times$ such that $\ell_a^*(q^e) = \lambda(q^e)$. But a is also a generator for $(\mathbb{Z}/q^f\mathbb{Z})^\times$, i.e. $\ell_a^*(q^f) = \lambda(q^f)$. By Dirichlet's Theorem, there exists a prime $p \equiv a \pmod{q^e}$. Thus, we can certainly find a prime p with $\ell_p^*(q^f) = \lambda(q^f)$ for all f with $0 \leq f \leq e$.

If $n = 2^e$, we observe that, when $p = 3$, we have $\ell_p^*(2^j) = \lambda(2^j)$ for all $j \geq 1$. Hence, $\ell_3^*(d) = \lambda(d)$ for all divisors d of 2^e.

Now we consider the case where $n = q_1^{e_1} \cdots q_k^{e_k}$, $k \geq 2$. Each $d \mid n$ can be written in the form $d = q_1^{f_1} \cdots q_k^{f_k}$, where $0 \leq f_i \leq e_i$ holds for $i = 1, ..., k$. For each i, if q_i is odd, let a_i be a primitive root (mod $q_i^{e_i}$), and if $q_i = 2$, take $a_i = 3$. Since $q_1, ..., q_k$ are pairwise relatively prime then, by the Chinese Remainder Theorem, there exists an integer x with

$$x \equiv a_1 \pmod{q_1^{e_1}}$$
$$\vdots$$
$$x \equiv a_k \pmod{q_k^{e_k}}.$$

By Dirichlet's Theorem, there exists a prime p with $p \equiv x \pmod{n}$. In other words, $\ell_p^*(q_i^{e_i}) = \lambda(q_i^{e_i})$ for $i = 1, ..., k$. As remarked earlier, we have $\ell_p^*(q_i^{f_i}) = \lambda(q_i^{f_i})$ for all f_i with $0 \leq f_i \leq e_i$. Therefore, since $q_1, ..., q_k$ are pairwise relatively prime, we have

$$\ell_p^*(q_1^{f_1} \cdots q_k^{f_k}) = \operatorname{lcm}[\ell_p^*(q_1^{f_1}), \cdots, \ell_p^*(q_k^{f_k})] = \operatorname{lcm}[\lambda(q_1^{f_1}), \cdots, \lambda(q_k^{f_k})] = \lambda(q_1^{f_1} \cdots q_k^{f_k}).$$

□

Below, we provide the proof of Theorem 3.1.

Proof. If n is λ-practical then, by Lemma 3.3, there exists a prime p' such that $\ell_{p'}^*(d) = \lambda(d)$ for all $d \mid n$. Since n is p-practical for all primes p then, in particular, n is p'-practical, i.e. for all integers m with $1 \leq m \leq n$, we have

$$m = \sum_{d \mid n} \ell_{p'}^*(d) n_{p'}(d),$$

where $n_{p'}(d)$ is an integer satisfying $0 \leq n_{p'}(d) \leq \frac{\varphi(d)}{\ell_{p'}^*(d)}$. Thus, for all m with $1 \leq m \leq n$, we have

$$m = \sum_{d \mid n} \lambda(d) n_{p'}(d),$$

since $\ell_{p'}^*(d) = \lambda(d)$ for all $d \mid n$. Since it is necessarily the case that $0 \leq n_{p'}(d) \leq \frac{\varphi(d)}{\ell_{p'}^*(d)} = \frac{\varphi(d)}{\lambda(d)}$, then n satisfies the condition given in Theorem 3.1.

On the other hand, suppose that every integer m with $1 \leq m \leq n$ can be written in the form $m = \sum_{d \mid n} \lambda(d) m_d$, where m_d is an integer satisfying $0 \leq m_d \leq \frac{\varphi d}{\lambda(d)}$. By definition, $\lambda(d) = \max_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} \ell_a^*(d)$. Since $\ell_a^*(d) \leq \lambda(d)$ for all a in $(\mathbb{Z}/d\mathbb{Z})^\times$, then certainly every m with $1 \leq m \leq n$ can be written in the form $m = \sum_{d \mid n} \ell_p^*(d) n_d$, where p is any rational prime and $0 \leq n_d \leq \frac{\varphi(d)}{\ell_p^*(d)}$. Thus, n is λ-practical. □

4. Key lemmas

In this section, we provide some key lemmas for characterizing the relationship between φ-practical and λ-practical numbers. These lemmas will be used in section 5 in order to obtain information about the relative asymptotic densities of these sets. We begin by reminding the reader of some useful results on the φ-practical numbers. In [8], we proved the following necessary condition for an integer n to be φ-practical:

Lemma 4.1. *Suppose that $n = p_1^{e_1} \cdots p_k^{e_k}$ is φ-practical, where $p_1 < p_2 < \cdots < p_k$ and $e_i \geq 1$ for $i = 1, ..., k$. Define $m_i = p_1^{e_1} \cdots p_i^{e_i}$ for $i = 0, ..., k-1$. Then, the inequality $p_{i+1} \leq m_i + 2$ must hold for all i.*

We say that an integer n is *weakly φ-practical* if all of its prime factors satisfy the inequality from Lemma 4.1. We note that the inequality in Lemma 4.1 also gives a necessary condition for a positive integer n to be λ-practical. Namely, if $p_{i+1} > m_i + 2$ for some i such that $0 \leq i \leq k-1$ then, since $\lambda(p_{i+1}) = \varphi(p_{i+1}) = p_{i+1} - 1$, we have $\lambda(p_{i+1}) > m_i + 1$. Since $m_i = \sum_{d|m_i} \lambda(d) \frac{\varphi(d)}{\lambda(d)}$, then $m_i + 1$ cannot be written as a sum of $\lambda(d)$'s, so such an n would not be λ-practical. Thus, we have proven the following:

Lemma 4.2. *Every λ-practical number is weakly φ-practical.*

The converse to Lemma 4.2 is false. For example, $n = 9$ is weakly φ-practical but not λ-practical. However, we can show that the converse holds for even integers and for squarefree integers. In order to complete these proofs, we will need a lemma on the structure of λ-practical numbers. The following result (cf. [8][Lemma 4.1]) gives a partial characterization for the structure of φ-practical numbers:

Lemma 4.3. *If M is φ-practical and p is prime with $(p, M) = 1$, then $M' = pM$ is φ-practical if and only if $p \leq M + 2$. Moreover, $M' = p^k M, k \geq 2$ is φ-practical if and only if $p \leq M + 1$.*

The statement of Lemma 4.3 mirrors the statement of Lemma 2.2, with one important difference: although Lemma 2.2 yields a necessary-and-sufficient condition for all numbers to be practical, Lemma 4.3 cannot be used to generate the full list of φ-practical numbers. For example, $3^2 \times 5 \times 17 \times 257 \times 65537 \times (2^{31} - 1)$ is φ-practical, but none of the numbers 3^2, $3^2 \times 5$, $3^2 \times 5 \times 17$, $3^2 \times 5 \times 17 \times 257$, or $3^2 \times 5 \times 17 \times 257 \times 65537$ are φ-practical. Nevertheless, Lemma 4.3 is useful in obtaining a lower bound for the number of φ-practical numbers up to X. Likewise, to find a lower bound for the number of λ-practical numbers that fail to be φ-practical (which we shall accomplish in Section 5), we will make use of the following lemma:

Lemma 4.4. *Let $n = mp$, where m is λ-practical, $p \leq m + 2$ and $(p, m) = 1$. Then n is λ-practical. Moreover, if $n = p^k m$ with $k \geq 2$, then n is λ-practical if $p \leq m + 1$.*

The proof of Lemma 4.4 is virtually identical to the proof of Lemma 4.3. The idea is to use the characterization of λ-practical numbers given in Theorem 3.1 to show that every integer $l \in [1, n]$ can be expressed in the form

$$l = \sum_{d|m} \lambda(d) m_d,$$

with $0 \leq m_d \leq \frac{\varphi(d)}{\lambda(d)}$. In order to check that this holds when $n = mp$, we observe that if every $l \in [1, n]$ can be written in the form

$$l = (p-1)Q + R, \ 0 \leq Q, R \leq m \tag{4.1}$$

then, using our hypothesis that m is λ-practical, we have

$$l = \sum_{d|m} (p-1)\lambda(d) m_d + \sum_{d|m} \lambda(d) m'_d, \tag{4.2}$$

where $0 \leq m_d, m'_d \leq \frac{\varphi(d)}{\lambda(d)}$. We can use the facts that $\lambda(p) = p - 1$ and $\lambda(p_1^{e_1} \cdots p_k^{e_k}) = \mathrm{lcm}[\lambda(p_1^{e_1}), ..., \lambda(p_k^{e_k})]$ to show that we can re-write (4.2) in the following manner:

$$l = \sum_{d|m} \lambda(pd) m_{pd} + \sum_{d|m} \lambda(d) m'_d,$$

where $0 \leq m_{pd} \leq \frac{\varphi(pd)}{\lambda(pd)}$ and $0 \leq m'_d \leq \frac{\varphi(d)}{\lambda(d)}$. Thus, the proof boils down to showing that every $l \in [1, n]$ can be expressed as in (4.1), which follows from breaking $[1, n]$ into subintervals of the form $[(p-1)Q, (p-1)Q + m]$ and using the hypothesis that $p \leq m + 2$ to show that the subintervals cover the full interval. The higher power case is similar, but requires induction on the power of the prime p.

Proposition 4.5. *Let n be an even integer. Then n is weakly φ-practical if and only if n is φ-practical if and only if n is λ-practical.*

Proof. We will begin by showing that an even integer n is weakly φ-practical if and only if it is φ-practical. If n is even and weakly φ-practical, then we can write $n = p_1^{e_1} \cdots p_k^{e_k}$, where $2 = p_1 < p_2 < \cdots < p_k$ and $e_i \geq 1$ for $i = 1, ..., k$. We will use induction on the number of distinct prime factors of n to show that n is φ-practical. For our base case, we observe that 2^{e_1} is φ-practical for all positive values of e_1. For our induction hypothesis, we assume that $m = p_1^{e_1} \cdots p_{k-1}^{e_{k-1}}$ is φ-practical. Since m is even and p_k is odd, then $p_k \leq m + 2$ implies that $p_k \leq m + 1$. Thus, by Lemma 4.3, $mp_k^{e_k}$ is φ-practical. The other direction follows immediately from Lemma 4.1. The proof for λ-practicals is the same, this time using Lemma 4.4 instead of Lemma 4.3. ∎

As we remarked above, the conditions given in Lemma 4.1 for an integer n to be weakly φ-practical are necessary, but not sufficient, for n to be φ-practical. However, when n is squarefree, we have shown (cf. Corollary 4.2 in [8]) that these notions are equivalent. There is an analogous situation for λ-practical numbers.

Proposition 4.6. *Let n be a squarefree integer. Then n is λ-practical if and only if it is φ-practical.*

Proof. We showed in Corollary 4.2 of [8] that a squarefree integer is φ-practical if and only if it is weakly φ-practical. From Lemma 4.2, every λ-practical number is weakly φ-practical. The other direction of the proof is trivial, as all φ-practical numbers are automatically λ-practical. □

While it is easy to see that all φ-practical numbers are λ-practical, the converse does not hold. In fact, it is easy to show that there are infinitely many counterexamples:

Proposition 4.7. *There are infinitely many λ-practical numbers that are not φ-practical.*

Proof. Let $X \geq 1$ be a real number. Let $n = 45 \cdot \prod_{23<p\leq X} p$. It follows from Bertrand's Postulate that every prime $p \mid n$ with $p \nmid 45$ satisfies $p \leq m + 2$, where m is the product of 45 and all of the primes $23 < q < p$. Then, since 45 is λ-practical, it follows from Lemma 4.4 that n is λ-practical. However, n is not φ-practical, since $x^{45} - 1$ has no divisor of degree 22 and all other primes $p \mid n$ are greater than 23, so $\lambda(p) > 22$. Thus, as we let X tend to infinity, we see that this method produces an infinite family of λ-practical numbers that are not φ-practical. □

We will elaborate on the ideas presented in the proof of Proposition 4.7 in the next section when we determine the asymptotic density of λ-practical numbers that fail to be φ-practical.

5. Proof of Theorem 1.1

In this section, we will discuss the distribution of λ-practical numbers in relation to that of the φ-practical numbers. We begin by reminding the reader of the method of proof in (1.1), which will be a model for some of the arguments that we will use to bound the number of λ-practical integers up to X. The key to proving the upper bound in (1.1) was to use Proposition 4.5 in order to show that all even φ-practical numbers are practical. To handle the case of odd φ-practicals, we observed that, for every odd integer n in $(0, X]$, there exists a unique positive integer l such that $2^l n$ is in the interval $(X, 2X]$. Moreover, we showed that $2^l n$ is φ-practical if n is φ-practical. As a result, we were able to construct a one-to-one map from the set of odd φ-practical numbers in $(1, X]$ to a subset of the even φ-practical numbers in $(X, 2X]$.This allowed us to directly compare the size of the set of φ-practical numbers with the size of the set of practical numbers, which we knew to be $O(X/\log X)$ from [3, Theorem 2].

We can use the same argument to show that the upper bound given in (1.1) will also serve as an upper bound for the number of λ-practical numbers up to X. The only modification needed is to use Proposition 4.5 to show that all even λ-practical numbers are practical. On the other hand, Lemma 4.2 shows that, if n is an odd λ-practical number, then it is weakly φ-practical. Thus, for $l \geq 1$, $2^l n$ is weakly φ-practical, since multiplying a weakly φ-practical integer n by a power of 2 will not prevent its prime divisors from satisfying the inequalities from Lemma 4.1. Therefore, we can use the argument given above for the odd φ-practicals to obtain the same upper bound for the number of λ-practicals up to X.

In order to obtain a lower bound, we simply observe that the set of φ-practical numbers is properly contained within the set of λ-practical numbers. Hence, the lower bound that we gave in [8] for the φ-practical numbers will also serve as a lower bound for the λ-practical numbers. As a result, we have:

Proposition 5.1. *Let $F_\lambda(X) = \#\{n \leq X : n$ is λ-practical$\}$. Then, there exist positive constants c_3 and c_4 such that*

$$c_3 \frac{X}{\log X} \leq F_\lambda(X) \leq c_4 \frac{X}{\log X},$$

for all $X \geq 2$.

Note that the argument above shows that we may, in fact, take $c_3 = c_1$ and $c_4 = c_2$ (where c_1 and c_2 are the constants from (1.1)). However, this does not imply that $F_\lambda(X) - F(X) = o(\frac{X}{\log X})$. In fact, as stated in Theorem 1.1, we can show that $F_\lambda(X) - F(X) \gg \frac{X}{\log X}$. Before we prove this result, we remind the reader of some definitions and lemmas used in the lower bound argument for the φ-practical numbers in [8], which will be useful in this scenario as well.

Let $1 = d_1(n) < d_2(n) < \cdots < d_{\tau(n)}(n) = n$ denote the increasing sequence of divisors of a positive integer n. We define

$$T(n) = \max_{1 \leq i < \tau(n)} \frac{d_{i+1}(n)}{d_i(n)}.$$

Definition 5.2. An integer n is called 2-*dense* if n is squarefree and $T(n) \leq 2$.

Note that any 2-dense number $n > 1$ is even. Let

$$D(X) = \#\{1 \leq n \leq X : n \text{ is 2-dense}\}.$$

In [3], Saias built on techniques developed by Tenenbaum (cf. [6], [7]) in order to prove the following upper and lower bounds for $D(X)$:

Lemma 5.3 (Saias). *There exist positive constants κ_1 and κ_2 such that*

$$\kappa_1 \frac{X}{\log X} \leq D(X) \leq \kappa_2 \frac{X}{\log X}$$

for all $X \geq 2$.

The lower bound for the 2-dense integers is also a lower bound for the count of practical numbers up to X, since every 2-dense integer automatically satisfies Stewart's condition. Unfortunately, we cannot extrapolate any information about a lower bound for $F(X)$ from the lower bound for $D(X)$ because the 2-dense integers are not necessarily φ-practical (for example, $n = 66$). In [8], we got around this problem by adopting the following modification on the definition of 2-dense:

Definition 5.4. A 2-dense number n is *strictly 2-dense* if $\frac{d_{i+1}}{d_i} < 2$ holds for all i satisfying $1 < i < \tau(n) - 1$.

We showed (cf. Lemma 5.4 in [8]) that the strictly 2-dense integers have an important relationship with the φ-practical numbers:

Lemma 5.5. *Every strictly 2-dense number is φ-practical.*

Recall that, in Proposition 4.7, we constructed an infinite family of λ-practical numbers that are not φ-practical. We will use this construction, along with several lemmas concerning the 2-dense and strictly 2-dense numbers, in order to prove the first of our main theorems.

Proof of Theorem 1.1. We begin by recalling an argument given in [8]. Let $n = mpj$, where m is a 2-dense integer, p is a prime satisfying $m < p < 2m$, and j is an integer that has the following properties: $j \leq X/mp$, $P^-(j) > p$ and mpj is 2-dense. Let $C > 16$ be an integer that is chosen to be large relative to the size of the constant κ_1 from Lemma 5.3. For each integer $k > C$, we consider those 2-dense numbers $m \in (2^{k-1}, 2^k]$. Since $m < p < 2m$, we must have $p \in (2^{k-1}, 2^{k+1})$. We say that n has an *obstruction at k* if m and p land within these intervals, i.e., if p is a prime in our construction that might prevent n from being strictly 2-dense. In Theorem 5.11 in [8], we showed that, if C is large enough, the number of 2-dense integers with obstructions at $k > C$ is negligible relative to the full count of 2-dense integers. Thus, consider the set

$$\mathcal{N} = \{n \leq X : n \text{ is 2-dense with no obstructions at } k > C\}.$$

For an appropriate choice of $C \geq 5$, we have

$$\#\mathcal{N} \geq \kappa \frac{X}{\log X},$$

where $\kappa > 0$ is some absolute constant. As in [8], we define a function $f : \mathcal{N} \to \mathbb{Z}_+$ to be a function that maps each element $n \in \mathcal{N}$ to its largest 2-dense divisor with all prime factors less than or equal to 2^C. Let $\mathcal{M} = \mathrm{Im} f$. The Pigeonhole Principle guarantees that there is some $m_0 \in \mathcal{M}$ that has at least

$$\frac{\#\mathcal{N}}{\#\mathcal{M}} \geq \frac{\kappa}{4^{2^C}} \frac{X}{\log X}$$

elements in its preimage, since the Chebyshev bound (cf. [2, pg. 108]) implies that $\prod_{p \leq 2^C} p \leq 4^{2^C}$. In other words, $m_0 = f(n)$ for at least the average number of integers in a pre-image.

Now, for each $n \in \mathcal{N}$ with $f^{-1}(m_0) = n$, let

$$n' = \frac{3 \cdot 5 \cdot 29^5}{\gcd(2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23, m_0)} \, n \prod_{\substack{29 < p \leq 2^C \\ p \text{ prime} \\ p \nmid m_0}} p. \tag{5.1}$$

Since n is 2-dense, it must be the case that $3 \mid n$, hence $3^2 \| n'$. Also, we must have $5 \| n'$, since if $5 \mid n$ then $5 \mid m_0$, so it is removed in the denominator of n'. Thus, the only 5 that appears in the factorization of n' is the one that appears in the numerator of (5.1). Now, n' does not have any other prime factors smaller than 29, since if n is divisible by a prime $q < 29$, then $q \mid m_0$, hence $q \mid \gcd(2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23, m_0)$. Thus, n' is not φ-practical, since the absence of small primes (aside from the divisors of 45) makes it so that $x^n - 1$ has no divisor of degree 22. However, we will show that n' is λ-practical. Let

$$l = n \prod_{\substack{29 < p \leq 2^C \\ p \text{ prime} \\ p \nmid m_0}} p.$$

Since n is 2-dense then $2 \mid n$. Moreover, if we enumerate the prime factors of l in increasing order, where p_i is the i^{th} smallest, then Bertrand's postulate implies that all of the primes p_i dividing l that are greater than 29 satisfy $p_{i+1} \leq 2p_i$. Thus, they satisfy the inequality given in Lemma 4.1 as well. Let

$$r = \frac{3 \cdot 5 \cdot 29^5}{\gcd(2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23, m_0)},$$

so $n' = l \cdot r$. Multiplying l by r does not prevent the primes greater than or equal to 29 from satisfying the inequality from Lemma 4.1, since $29^5 > \frac{1}{3} \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, so that $r > 1$. In other words, if a prime factor p of n' satisfies $p \leq m + 2$, then it is certainly the case that $p \leq mr + 2$. Thus, we have just shown that n' has the following structure: $n' = 45M$, where $P^-(M) = 29$ and all of the prime factors of $45M$ satisfy the inequality from Lemma 4.1. Since 45 is λ-practical, then Lemma 4.4 implies that n' is λ-practical. Now, since we multiplied every n in the pre-image of m_0 by the same number, there is a one-to-one correspondence between λ-practical numbers up to $r \cdot 4^{2^C} X$ that we have constructed and the 2-dense numbers in the pre-image of m_0. As a result, at least $\frac{\kappa}{4^{2^C}} \frac{X}{\log X}$ of the integers up to $r \cdot 4^{2^C} X$ are λ-practical. □

Note: By Lemma 4.2, all of the λ-practical numbers that we have constructed in the proof of Theorem 1.1 are weakly φ-practical. As a result, this argument also shows that, for X sufficiently large, there are $\gg \frac{X}{\log X}$ weakly φ-practicals in $[1, X]$ that are not φ-practical.

6. Proof of Theorem 1.2

Our next endeavor will be to quantify the p-practical numbers that fail to be λ-practical. We begin with the following analogue of Lemma 4.2, which is proven in the same manner as its predecessor.

Lemma 6.1. *Let $n = mq$, where m is p-practical and q is a prime satisfying $\ell_p^*(q) \leq m + 1$, with $(q, m) = 1$. Then n is p-practical. Moreover, if $n = mq^k$ where $k \geq 2$, then n is p-practical if $\ell_p^*(q) \leq m$.*

We can use Lemma 6.1 to prove our first result comparing the sizes of the sets of λ-practical and p-practical numbers.

Proposition 6.2. *For each prime p, there are infinitely many p-practicals that are not λ-practical.*

Proof. Case 1: If $p=2$, let $n = 21 \cdot \prod_{7<q\leq X} q$. Since 21 is 2-practical and since each prime q_0 in the product over q satisfies the inequality $q_0 \leq \prod_{7<q<q_0} q + 2$, then n is 2-practical by Lemma 6.1. However, n is not λ-practical, since we cannot write 4 in the form described in Theorem 3.1. Thus, by letting X tend to infinity, we see that this method will generate an infinite family of 2-practical numbers that are not λ-practical.

Case 2: For each prime $p \geq 3$,we will show that there exists a prime $q_0 \neq 3$ dividing (p^2+p+1) and, for this q_0, the number $2q_0$ is p-practical but not λ-practical. First, observe that if such a prime exists, then $q_0 \mid (p^2+p+1)$ implies that $p^3 \equiv 1 \pmod{q_0}$, i.e. $\ell_p^*(q_0) \leq 3$. Thus, since $m_0 = 2$ is p-practical for all primes p and $\ell_p^*(q_0) \leq m_0+1$, then $2q_0$ is p-practical by Lemma 6.1. However, $2q_0$ is not λ-practical, since $q_0 > 3$ implies that $\lambda(q_0) \geq 4$. Thus, we cannot write 3 in the form described in Theorem 3.1.

Now, we will prove the existence of a prime q_0 satisfying the conditions given above. The argument boils down to proving that p^2+p+1 is not a power of 3. In the case where $p=3$, we have $p^2+p+1 = 13$. Suppose that $p > 3$. Then, it must be the case that $p \equiv \pm 1 \pmod 3$. If $p \equiv -1 \pmod 3$, then $p^2+p+1 \equiv 1 \pmod 3$, so p^2+p+1 is not divisible by 3. On the other hand, if $p \equiv 1 \pmod 3$ then, if p^2+p+1 were a power of 3, the fact that $p > 3$ forces p^2+p+1 to be divisible by 9. However, the congruence $x^2+x+1 \equiv 0 \pmod 9$ has no solutions. □

The infinite families that we have just constructed will play an important role in the proof of Theorem 1.2, which we will present at the end of this section. We remark that we could also have proven the second case in Proposition 6.2 using the following lemma:

Lemma 6.3. *If $n = p^k$ with $k \geq 0$, then n is p-practical.*

Proof. Let $n = p^k$, with $k \geq 0$. Using the binomial theorem, we have $x^{p^k} - 1 = (x-1)^{p^k}$ in $\mathbb{F}_p[x]$. Hence, $x^n - 1$ has a divisor of every degree, so n is p-practical. □

In order to generate an infinite family of p-practical numbers when $p \geq 5$, we could simply have taken $n = p^k$, where k ranges over all positive integers. By Lemma 6.3, n is p-practical. However, when p is in this range, we have $\lambda(p) = p-1 \geq 4$. In other words, the gap between 1 and $\lambda(p)$ is too large for p^k to be λ-practical. In the case where $p=3$, we can take $n = p^k$ with $k \geq 2$. Then 4 cannot be written in the form described in Theorem 3.1, so n fails to be λ-practical.

Proof of Theorem 1.2. This proof is nearly identical to the proof of Theorem 1.1. The main difference arises in our construction of n', which varies depending on our choice of p. If $p = 2$, we let

$$n' = \frac{7^2}{\gcd(10, m_0)}\, n \prod_{\substack{7 < q \le 2^C \\ q \nmid m_0}} q,$$

where C, m_0 and n are defined as in the proof of Theorem 1.1. Then n' is of the form $n' = 21 \cdot m$, where $P^-(m) \ge 7$ and the primes dividing m satisfy the weakly φ-practical conditions. As we showed in the proof of Lemma 6.2, 21 is 2-practical but not λ-practical. Thus, since all of the prime factors of m are at least 7, it follows from Lemma 4.4 that n' is not λ-practical. To show that n' is 2-practical, we use Lemma 6.1 in place of Lemma 4.2 in the proof of Theorem 1.1.

The arguments for $p \ge 3$ follow the same line of reasoning. In the case where $p = 3$, we define

$$n' = \frac{13^4}{\gcd(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, m_0)}\, n \prod_{\substack{13 < q \le 2^C \\ q \nmid m_0}} q.$$

Then n' is of the form $n' = 26 \cdot 13^3 \cdot m$, where $P^-(m) > 13$ and all of its prime factors satisfy the weakly φ-practical conditions. Since 26 is 3-practical but not λ-practical, we can use the same reasoning from the $p = 2$ case to show that n' is indeed 3-practical but not λ-practical. In the case where $p \ge 5$, we let

$$n' = \frac{2p^2}{\gcd(6p, m_0)}\, n \prod_{\substack{p < q \le 2^C \\ q \nmid m_0}} q. \tag{6.1}$$

Since m_0 is 2-dense, it must be the case that m_0 is divisible by 6 and by at least one of the primes 5 and 7. In other words, there are three cases that we need to consider: If $42 \mid m_0$, then (6.1) yields $n' = 14p^2m$. If $30 \mid m_0$ but $7 \nmid m_0$, we have $n' = 10p^2m$. If $210 \mid m_0$, then $n' = 70p^2m$. Either way, n' is not λ-practical, since $10, 14$ and 70 are not λ-practical and $P^-(m) > p \ge 5$. As in the cases above, we can use the proof of Theorem 1.1 with Lemma 6.1 to show that n' is p-practical. □

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