Sums of proper divisors with missing digits

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The $s(n)$ function

Definition

Let $s(n)$ denote the sum of proper divisors of n.

Example: $s(p) = 1$ for any prime p

Example: $s(12) = 1 + 2 + 3 + 4 + 6 = 16$

We can write $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum-of-divisors function.

Perfect numbers

Pythagoras observed:

$$s(\mathbf{6}) = 1 + 2 + 3 = \mathbf{6}.$$

Definition

n is **perfect** if $s(n) = n$.

Amicable pairs

Definition

If $s(\mathbf{n}) = \mathbf{m}$, $s(\mathbf{m}) = \mathbf{n}$, and $\mathbf{m} \neq \mathbf{n}$, then $\mathbf{n}$ and $\mathbf{m}$ form an **amicable pair**.

Example (Pythagoras):

$$s(220) = 284, \quad s(284) = 220.$$

Motivating questions

"Studying the comparison of $s(n)$ to n led to theorems of Schoenberg, Davenport, and Erdős-Wintner, and the birth of probabilistic number theory." -Carl Pomerance

In this talk, we will focus on two particular questions concerning the function $s(n)$:

1. Which numbers are of the form $s(n)$?
2. How large is the set $s^{-1}(n)$?

And then we will involve the integers with missing digits...

The image of s

Erdős was the first to consider questions about the image of s.

Odd integers in the image of s

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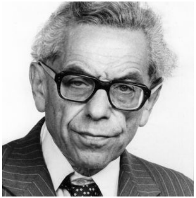
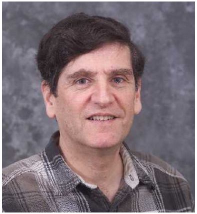
So almost all odd numbers ≥ 9 are values of s.

What about even numbers?





Theorem (Erdős, 1973)

A positive proportion of even integers are missing from the image of s.

Theorem (Luca & Pomerance, 2014)

A positive proportion of even integers are in the image of s.

The image of s

The function s can map sets of asymptotic density 0 to sets with positive asymptotic density.

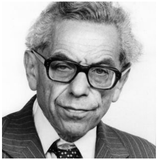
Example If $\mathcal{A} = \{pq : p, q \text{ prime}\}$ then $\mathcal{A}$ has asymptotic density 0 but $s(\mathcal{A})$ has asymptotic density $1/2$.

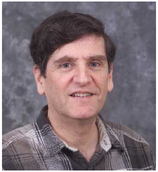
Example Erdős constructed sets $\mathcal{A}$ of positive density such that $s^{-1}(\mathcal{A})$ not only has density 0 but is, in fact, empty.

The preimage of s

What can be said about $s^{-1}(\mathcal{A})$ when $\mathcal{A}$ has asymptotic density 0?

The EGPS Conjecture





Conjecture (Erdős, Granville, Pomerance, Spiro, 1990)

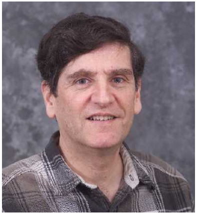
Let $\mathcal{A}$ be a set with asymptotic density 0. Then $s^{-1}(\mathcal{A})$ also has asymptotic density 0.

Special cases of EGPS

Some special cases of EGPS have been proven:

- (Pollack, 2014) If $\mathcal{A}$ is the set of primes, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.
- (Troupe, 2015)
  If $\mathcal{A}_\epsilon = \{m : |\omega(m) - \log\log m| > \epsilon \log\log m\}$ then $s^{-1}(\mathcal{A}_\epsilon)$ has asymptotic density 0.
- (Pollack, 2015) If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.
- (Troupe, 2020) If $\mathcal{A}$ is the set of integers that can be written as a sum of two squares, then $s^{-1}(\mathcal{A})$ has asymptotic density 0.

Partial Progress on EGPS



Theorem (Pollack, Pomerance, T., 2017)

Let $\epsilon \to 0$ as $x \to \infty$. Suppose $\mathcal{A}$ is a set of at most $x^{1/2+\epsilon}$ positive integers. Then, as $x \to \infty$,

$$\#\{n \le x : s(n) \in \mathcal{A}\} = o_\epsilon(x)$$

uniformly in $\mathcal{A}$.

Consequences

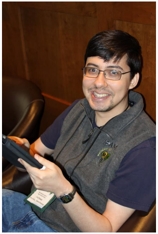
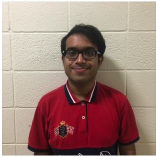
Immediate consequences of our result:

- If $\mathcal{A}$ is the set of palindromes in any given base, then $s^{-1}(\mathcal{A})$ has density 0.
- If $\mathcal{A}$ is the set of squares, then $s^{-1}(\mathcal{A})$ has density 0.

Other recent related problems

Some very recent progress on $s(n)$:

- (Pollack and Singha Roy, 2022) For any fixed $k \geq 4$, the k-th power-free values of n and $s(n)$ are equally common.
- (Lebowitz-Lockard, Pollack, Singha Roy, 2023) The values of $s(n)$ (for composite n) are equidistributed among the residue classes modulo p for small primes p.
- (Pollack and Troupe, 2023) The function $\omega(s(n))$ has the same mean and variance as $\omega(n)$.


Lebowitz-Lockard, Pollack, Singha Roy, and Troupe

Integers with missing digits

Defining integers with restricted digits

For a proper subset $\mathcal{D} \subsetneq \{0, \ldots, g-1\}$ such that $0 \in \mathcal{D}$, we define

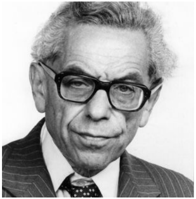
$$\mathcal{W}_{\mathcal{D}} := \left\{ n \in \mathbb{N} : n = \sum_{j \geq 0} \varepsilon_j(n) g^j, \varepsilon_j(n) \in \mathcal{D} \right\}$$

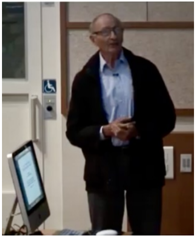
and

$$\mathcal{W}_{\mathcal{D}}(x) := \mathcal{W}_{\mathcal{D}} \cap [1, x]\,.$$

Notice that this set has asymptotic density 0.

Early results on integers with missing digits





Theorem (Erdős, Mauduit, and Sárközy, 1998)

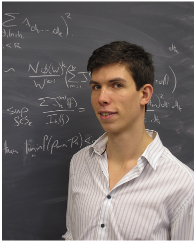
Integers with missing digits are well-distributed in arithmetic progressions.

Some of my co-author's work

Theorem (Dartyge and Mauduit, 2000)

There exist infinitely many $n \in \mathcal{W}_{\{0,1\}}$ *with at most* $(1+o(1))8g/\pi$ *prime factors as* $g \to \infty$.

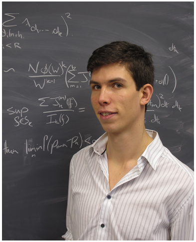
Primes with missing digits



Theorem (Maynard, 2019)

There are infinitely many primes with missing digits.

Polynomial values with missing digits



Theorem (Maynard, 2022)

There are infinitely many n *such that* $P(n) \in \mathcal{W}_{\mathcal{D}}$*, for any given non-constant polynomial* $P \in \mathbb{Z}[X]$*, large enough base* g*, and* $\mathcal{D} = \{0, \dots, g-1\} \setminus \{a_0\}$*.*

Our WINE Project

Our WINE Project: we study $\mathcal{W}_{s,\mathcal{D}} := s^{-1}(\mathcal{W}_{\mathcal{D}})$.

Sums of proper divisors with missing digits

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2023)

Let $\mathcal{A}$ be a set of integers with missing digits in any base $g \geq 3$. Then $s^{-1}(\mathcal{A})$ has asymptotic density 0.

In other words, the EGPS Conjecture holds for sets of integers with missing digits!

An effective result

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2023)

Let $g \geq 3$, $\mathcal{D} \subsetneq \{0, ..., g-1\}$ be a nonempty subset. Then for all x sufficiently large,

$$\#\mathcal{W}_{s,\mathcal{D}}(x) = \#s^{-1}(\mathcal{W}_{\mathcal{D}}(x)) \leq \frac{x}{\exp((\log_2 x)^{1+o(1)})}.$$

How sharp is our bound?

Recall that $s(p) = 1$ for all primes p.

Then, if $\mathcal{D}$ contains 1, it follows that

$$\#\mathcal{W}_{s,\mathcal{D}}(x) \geq \pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

Thus, our exponent of $1 + o(1)$ is optimal for arbitrary $g, \mathcal{D}$.

An application of Maynard's work

Theorem (Benli, Cesana, Dartyge, Dombrowsky, T., 2023)

The function $s(n)$ takes infinitely many values in $\mathcal{W}_{\mathcal{D}}$.

Merci!