Counting and effective rigidity in algebra and geometry

Lola Thompson

Oberlin College

June 13, 2018

Collaborators

Joint work with: Ben Linowitz (Oberlin), Ben McReynolds (Purdue), and Paul Pollack (UGA).

Overview

Our Goal: Prove effective versions of rigidity results for arithmetic hyperbolic 2- and 3-manifolds.

Main Idea: We exploit the correspondence between maximal subfields of quaternion algebras and lengths of geodesics on arithmetic hyperbolic 2- and 3-manifolds.

Talk Overview:

- Brief introduction to quaternion algebras
- Our results on counting quaternion algebras
- Geometric background
- How to construct surfaces from quaternion algebras
- Our results on surfaces

A brief introduction to quaternion algebras and orders

Quaternion algebras and orders

In the 1830s and 1840s William Rowan Hamilton sought a number system which would play a role in three-dimensional geometry analogous to that of the complex numbers in two-dimensional geometry.

"Every morning in the early part of the above-cited month [October 1843], on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them."'

– Hamilton (in a letter to his son)

Quaternion algebras and orders

Theorem (Hamilton, 1843)

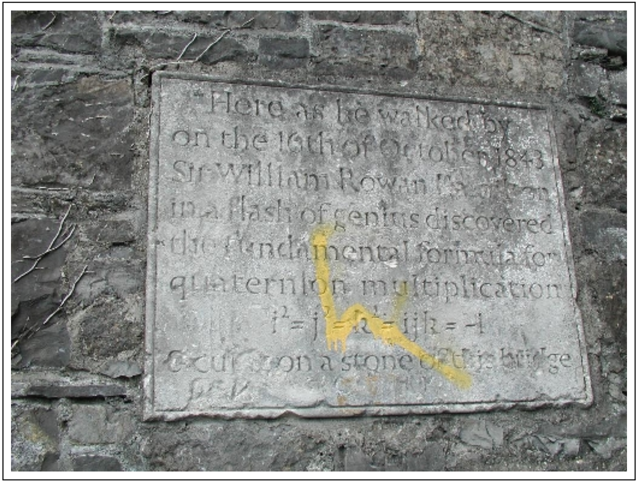
The $\mathbb{R}$-algebra $\mathbb{H}$ with basis $\{1, i, j, ij\}$ and defining relations

$$i^2 = -1 \qquad j^2 = -1 \qquad ij = -ji$$

is a four-dimensional division algebra.

Hamilton was so excited by this discovery that he carved these relations into the stone of the Brougham Bridge!

Quaternion algebras and orders



Quaternion algebras and orders

Write $(-1, -1, \mathbb{R})$ instead of $\mathbb{H}$.

This notation suggests a number of ways to generalize $\mathbb{H}$.

For example, let $(1, 1, \mathbb{R})$ be the $\mathbb{R}$-algebra with basis $\{1, i, j, ij\}$ and defining relations

$$i^2 = 1 \qquad j^2 = 1 \qquad ij = -ji.$$

Then $(1, 1, \mathbb{R}) \cong \mathrm{M}_2(\mathbb{R})$ via $i \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $j \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Quaternion algebras and orders

More generally, we can define $(a, b, \mathbb{R})$ to be the $\mathbb{R}$-algebra with basis $\{1, i, j, ij\}$ and defining relations

$$i^2 = a \qquad j^2 = b \qquad ij = -ji \qquad a, b \in \mathbb{R}^*.$$

It's not too hard to show that

- $(a, b, \mathbb{R}) \cong \mathbb{H}$ if $a, b < 0$ and
- $(a, b, \mathbb{R}) \cong \mathrm{M}_2(\mathbb{R})$ otherwise.

Thus $(a, b, \mathbb{R})$ is either a division algebra or isomorphic to $\mathrm{M}_2(\mathbb{R})$.

Quaternion algebras and orders

There are other ways that we could have generalized $(-1,-1,\mathbb{R})$.

If F is a field of characteristic zero and $a, b \in F^*$ we can define the **generalized quaternion algebra** (a, b, F).

Example: Let $F = \mathbb{Q}$ and consider the $\mathbb{Q}$-algebra $(-1,-1,\mathbb{Q})$.

- Since $(-1,-1,\mathbb{Q}) \subsetneq (-1,-1,\mathbb{R})$, we see that $(-1,-1,\mathbb{Q})$ is a division algebra.
- As before we also see that $(1,1,\mathbb{Q}) \cong \mathrm{M}_2(\mathbb{Q})$.

Quaternion algebras and orders

Recall that if $F = \mathbb{R}$, then (a, b, F) is either a division algebra or it is isomorphic to $\mathrm{M}_2(\mathbb{R})$.

Theorem (Wedderburn)

For any field F, if the F-algebra (a, b, F) is not a division algebra then $(a, b, F) \cong \mathrm{M}_2(F)$.

Note that the case $F = \mathbb{R}$ is special – in general, there will not be a unique quaternion division algebra over F!

Quaternion algebras and orders

Extension of scalars:

Let F be a field and F'/F a field extension.

If A is a quaternion algebra over F then we can consider the quaternion algebra $A \otimes_F F'$ over F'.

Concretely, if $A = (a, b, F)$ then $A \otimes_F F' = (a, b, F')$.

This is especially important in arithmetic applications.

Quaternion algebras and orders

Reduced norm:

Let A be a quaternion algebra over a number field F.

Over the complex numbers, there is only one embedding $A \hookrightarrow \mathrm{M}_2(\mathbb{C})$.

The **reduced norm** of A is the composite map

$$A \hookrightarrow \mathrm{M}_2(\mathbb{C}) \stackrel{\det}{\to} \mathbb{C}$$

For $A = \mathrm{M}_2(F)$ the reduced norm coincides with the determinant.

Quaternion algebras and orders

Let F be a number field with ring of integers $\mathcal{O}_F$.

An **order** of an F-algebra is a subring which is also a finitely generated $\mathcal{O}_F$-module containing an F-basis of the algebra.

Example 1: $\mathbb{Z}[i]$ is a quadratic order of the $\mathbb{Q}$-algebra $\mathbb{Q}(i)$.

Example 2: $\mathrm{M}_2(\mathbb{Z})$ is a maximal order of $\mathrm{M}_2(\mathbb{Q})$.

Example 3: $\mathcal{O}_F[i,j]$ is always an order of the F-algebra (a, b, F) when $a, b \in \mathcal{O}_F$.

Our Algebraic Results

Counting Central Division Algebras

Let k be a number field and n a positive integer. We are interested in counting the number of central division algebras of dimension n^2 over k which have (norm of) discriminant less than x.

For example, if $k = \mathbb{Q}$ and $n = 2$, we are counting the number of **rational quaternion algebras** with discriminant less than x.

Our main tool in counting central division algebras will be the following Tauberian theorem of Delange.

Delange's Tauberian Theorem

Theorem (Delange's Tauberian Theorem)

Let $G(s) = \sum \frac{a_N}{N^s}$ be a Dirichlet series satisfying:

1. $a_N \geq 0$ *for all* N *and* $G(s)$ *converges for* $\mathfrak{Re}(s) > \rho$.
2. $G(s)$ *can be continued to an analytic function in the closed half-plane* $\mathfrak{Re}(s) \geq \rho$ *except possibly for a singularity at* $s = \rho$.
3. *There is an open neighborhood of* ρ *and functions* $A(s), B(s)$ *analytic at* $s = \rho$ *with* $G(s) = A(s)/(s-\rho)^\beta + B(s)$ *at every point in this neighborhood having* $\mathfrak{Re}(s) > \rho$.

Then as $x \to \infty$ *we have*

$$\sum_{N \leq x} a_N = \left(\frac{A(\rho)}{\rho\Gamma(\beta)} + o(1) \right) x^\rho \log(x)^{\beta-1}.$$

Growth Rate of Division Algebras

Theorem (Linowitz, McReynolds, Pollack, T.)

If $N(x)$ denotes the number of division algebras of dimension n^2 over k with $|\mathrm{disc}| < x$ and ℓ is the smallest prime divisor of n, then there is a constant $\delta_n > 0$ so that

$$N(x) = (\delta_n + o(1))x^{\frac{1}{n^2(1-1/\ell)}}(\log x)^{\ell-2},$$

as $x \to \infty$.

Proof Sketch

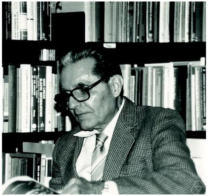
Main idea:

- Create a Dirichlet series whose coefficients count the central division algebras with fixed discriminant.

Proof Sketch

Main idea:

- Create a Dirichlet series whose coefficients count the central division algebras with fixed discriminant.
- Apply Delange's Theorem in order to estimate the partial sums of these coefficients.



Growth Rate of Algebras with a Specified Subfield

Theorem (Linowitz, McReynolds, Pollack, T.)

Fix a number field k and a quaternion algebra B defined over k. The number of quadratic extensions L/k which embed into B and satisfy $|\Delta_{L/k}| \leq x$ is asymptotic to $c_{k,B}x$, as $x \to \infty$, where $c_{k,B} > 0$. Moreover, if κ_k is the residue at $s = 1$ of $\zeta_k(s)$, r_2 is the number of pairs of complex embeddings of k, and r_B is the number of places of k (both finite and infinite) that ramify in B, then

$$c_{k,B} \geq \frac{1}{2^{r_B+r_2}} \frac{\kappa_k}{\zeta_k(2)}.$$

Proof Sketch

- Useful fact: If B/k is a quaternion algebra and L/k is a quadratic extension, then L embeds into B iff no prime of k that divides the discriminant of B splits in L/k.

Proof Sketch

- Useful fact: If B/k is a quaternion algebra and L/k is a quadratic extension, then L embeds into B iff no prime of k that divides the discriminant of B splits in L/k.
- A result of Matchett Wood allows us to model the splitting of finitely many primes as mutually independent events over the class of random quadratic extensions of k.

Proof Sketch

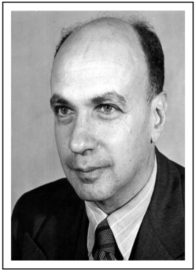
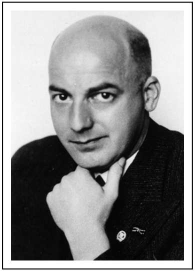
- Useful fact: If B/k is a quaternion algebra and L/k is a quadratic extension, then L embeds into B iff no prime of k that divides the discriminant of B splits in L/k.

- A result of Matchett Wood allows us to model the splitting of finitely many primes as mutually independent events over the class of random quadratic extensions of k.

- Use this along with the proof of the previous theorem.

Counting Quaternion Algebras

A similar problem is to count the number of quaternion algebras over k which admit embeddings of specified quadratic extensions $L_1, \ldots, L_r$ of k.


Theorem (Albert-Brauer-Hasse-Noether, 1931)

There is an embedding of L into A if and only if no prime $\mathfrak{p}$ of K for which $A \otimes_k k_{\mathfrak{p}}$ is a division algebra splits in L/k.

Counting Quaternion Algebras

We will assume that $[L_1 \cdots L_r : k] = 2^r$. Without this assumption it is possible that no quaternion division algebra will admit embeddings of all of the L_i.

Example: Let $k = \mathbb{Q}$ and consider the collection of quadratic extensions $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{10}), \mathbb{Q}(\sqrt{17})$. Every finite prime of $\mathbb{Q}$ splits in one of these extensions, so the discriminant of a quaternion algebra admitting embeddings of all of these extensions is divisible by no finite primes. The only such quaternion algebra is $M_2(\mathbb{Q})$, which is not a division algebra.

Counting Quaternion Algebras

The idea used to prove the previous theorems can be adapted to show:

Theorem (Linowitz, McReynolds, Pollack, T.)

Fix a number field k, and fix quadratic extensions $L_1, L_2, \ldots, L_r$ of k. Let L be the compositum of the L_i, and suppose that $[L : k] = 2^r$. The number of quaternion algebras over k with discriminant having norm less than x and which admit embeddings of all of the L_i is

$$\sim \delta \cdot x^{1/2}/(\log x)^{1-\frac{1}{2^r}},$$

as $x \to \infty$. Here δ is a positive constant depending only on the L_i and k.

Geometric Background

Geometric Background

Let M be a compact Riemannian manifold.

The **Laplace eigenvalue spectrum** of M, denoted $\mathcal{E}(M)$, is the multiset of eigenvalues of the Laplacian of M.

The **geodesic length spectrum**, denoted $LS(M)$, is the multiset of lengths of closed geodesics on M with fundamental group $\pi_1(M)$.

Inverse spectral geometry asks for the extent to which the spectra of M determine its geometry and topology.

Geometric terminology

Definition

Two manifolds are **commensurable** if and only if they have a common finite degree covering space.

Definition

Two manifolds are **isometric** if there is an isometry between them.

Definition

M and N are **isospectral** if $\mathcal{E}(M) = \mathcal{E}(N)$, and **length isospectral** if $LS(M) = LS(N)$.

Inverse Problems

Some natural inverse problems:

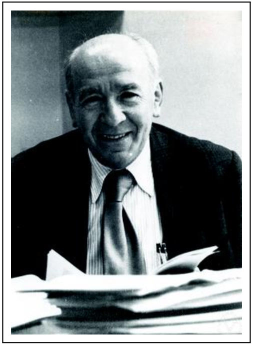
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, is M isometric to N?
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, are M and N commensurable?
- If $\mathrm{LS}(M) = \mathrm{LS}(N)$, what can be said about N?

We can ask the same questions for $\mathcal{E}(M)$.

Can you hear the shape of a drum?

Leon Green (1960) asked if the spectrum of M determines its isometry class.

The spectrum of M is essentially the collection of frequencies produced by a drumhead shaped like M.



Mark Kac (1966) popularized this question for planar domains:

Can you hear the shape of a drum?

Can you hear the shape of a drum?

Milnor had already shown that isometry class is not, in general, a spectral invariant.

Theorem (Milnor, 1964)

There exist lattices $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{16}$ such that the tori $\mathbb{R}^{16}/\Gamma_1$ and $\mathbb{R}^{16}/\Gamma_2$ are isospectral but not isometric.

Can you hear the shape of a drum?

Kac's question was finally answered in 1992.

Theorem (Gordon, Webb, Wolpert, 1992)

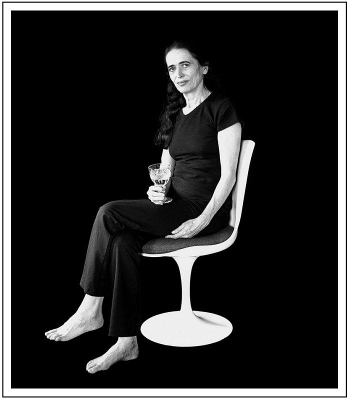
One cannot hear the shape of a drum.

Hyperbolic surfaces

The *hyperbolic plane* $\mathbf{H}^2$ is a simply connected surface with constant curvature -1 and can be be modeled by the disc (*Circle Limit IV*, by M.C. Escher):

The symmetries of this diagram, mapping one angel to any other angel, form a group of isometries acting on $\mathbf{H}^2$.

Results for hyperbolic surfaces

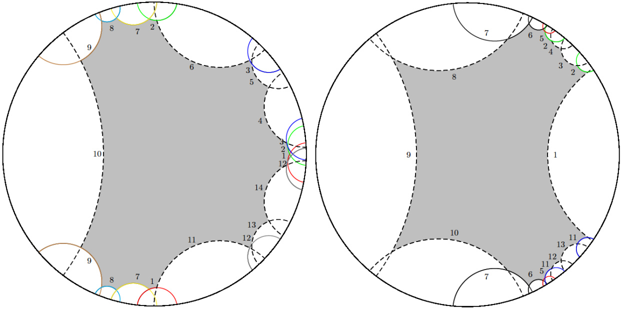


Theorem (Vigneras, 1980)

There exist isospectral non-isometric hyperbolic 2- and 3-manifolds.

Vigneras' examples arise from orders in quaternion algebras!

Isospectral but non-isometric



A pair of isospectral but non-isometric hyperbolic 2-orbifolds (due to B. Linowitz and J. Voight).

Constructing arithmetic manifolds

Elementary results from geometric group theory:

- $Isom^+(\mathbf{H}^2) \cong PSL_2(\mathbb{R})$.
- Every orientable hyperbolic 2-manifold is of the form $\mathbf{H}^2/\Gamma$ for some discrete subgroup Γ of $PSL_2(\mathbb{R})$.

We want to generalize the following construction of $PSL_2(\mathbb{Z})$:

$$M_2(\mathbb{Q}) \supset M_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}) \longrightarrow PSL_2(\mathbb{Z}).$$

We can replace $M_2(\mathbb{Q})$ with a *quaternion algebra* over a number field and $M_2(\mathbb{Z})$ with a *quaternion order*. Manifolds that arise in this manner are called *arithmetic manifolds*.

Constructing arithmetic manifolds

Let F be a totally real number field.

Let B/F be a quaternion algebra in which a unique real prime splits.

Let $\mathcal{O}$ be a maximal order of B.

Consider the embedding $\rho : B \to M_2(\mathbb{R})$.

Restricting ρ to the group $\mathcal{O}^1$ of elements of $\mathcal{O}$ with reduced norm 1 and projecting onto $PSL_2(\mathbb{R})$ gives an embedding

$$\bar{\rho} : \mathcal{O}^1 \to PSL_2(\mathbb{R}).$$

Constructing arithmetic manifolds

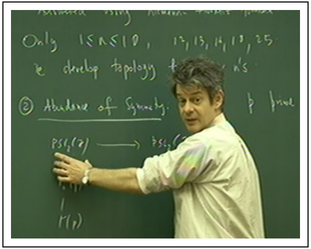
$\bar{\rho}(\mathcal{O}^1)$ is a discrete subgroup of isometries with finite covolume.

If B is a division algebra then $\bar{\rho}(\mathcal{O}^1)$ is cocompact.

If $\bar{\rho}(\mathcal{O}^1)$ is torsion-free then $\mathbf{H}^2/\bar{\rho}(\mathcal{O}^1)$ is a hyperbolic 2-manifold. (This amounts to saying that no cyclotomic field embeds into B.)

Hyperbolic surfaces commensurable with things of this form are defined to be *arithmetic*.

Results for arithmetic, hyperbolic surfaces



Theorem (Reid, 1992)

If M is an arithmetic, hyperbolic surface and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

Lubotzky, Samuels and Vishne (2005) showed: if one considers the symmetric space of $\mathrm{PGL}_n(\mathbb{R})$ or $\mathrm{PGL}_n(\mathbb{C})$ then, for $n > 3$, Reid's result is false and one can obtain arbitrarily large families of isospectral yet non-commensurable arithmetic manifolds.

Results for arithmetic, hyperbolic 3-manifolds

In the context of hyperbolic 3-manifolds, Reid's result is due to Chinburg, Hamilton, Long and Reid.

Theorem (Chinburg, Hamilton, Long, Reid, 2008)

If M and N are arithmetic hyperbolic 3-manifolds and $\mathrm{LS}(M) = \mathrm{LS}(N)$ then M and N are commensurable.

The length spectrum of hyperbolic 3-manifolds

Theorem (Futer and Millichap, 2016)

For every sufficiently large $n > 0$ there exists a pair of non-isometric finite-volume hyperbolic 3-manifolds $\{N_n, N_n^\mu\}$ such that:

1. $\mathrm{vol}(N_n) = \mathrm{vol}(N_n^\mu)$, *where this volume grows coarsely linearly with* n.
2. *The (complex) length spectra of* N_n *and* N_n^μ *agree up to length* n.
3. N_n *and* N_n^μ *have at least* e^n/n *closed geodesics up to length* n.
4. N_n *and* N_n^μ *are not commensurable.*

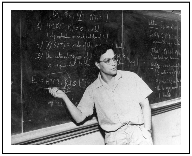
Our Geometric Results

Motivating Questions:

- Can we make Reid's result effective?
- How quickly does the number of commensurability classes of arithmetic, hyperbolic 2- or 3-manifolds grow?
- If the length spectra have a great deal of overlap, must the corresponding arithmetic, hyperbolic 2- or 3-manifolds be commensurable?

Effective Rigidity

To describe what we mean by **effective rigidity** some arithmetic facts will be useful.



Borel's finiteness result. For each $V \in \mathbb{R}_{\geq 0}$ there are only finitely many arithmetic hyperbolic 2- or 3-manifolds of volume at most V.

A consequence of Borel's result is that there exists $\mathrm{L}(V) \in \mathbb{R}_{\geq 0}$ such that if M and N are arithmetic surfaces of area at most V and have the same geodesic lengths up to $\mathrm{L}(V)$ then M and N are commensurable. The same result holds for 3-manifolds.

Effective Rigidity Results

The following are effective versions of Reid's "isospectral implies commensurable" result.

Theorem (Linowitz, McReynolds, Pollack, T.)

If M is an arithmetic hyperbolic surface then

$$L(V) \leq c_1 e^{c_2 \log(V) V^{130}}$$

for absolute, effectively computable constants c_1 and c_2.

Theorem (Linowitz, McReynolds, Pollack, T.)

If M is an arithmetic hyperbolic 3-manifold then

$$L(V) \leq c_3 e^{\log(V)^{\log(V)}}$$

where c_3 is an absolute, effectively computable constant.

Counting Arithmetic Manifolds

Let M be an arithmetic hyperbolic surface with fundamental group $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ and invariant trace field and quaternion algebra (K, B).

The closed geodesics

$$c_\gamma \colon S^1 \longrightarrow M$$

on M are in bijection with the Γ–conjugacy classes $[\gamma]_\Gamma$ of hyperbolic elements γ in Γ.

The associated geodesic length $\ell(c_\gamma)$ is given by

$$\cosh\left(\frac{\ell(c_\gamma)}{2}\right) = \pm\frac{\mathrm{Tr}(\gamma)}{2}.$$

Counting Arithmetic Manifolds

We denote by λ_γ the unique eigenvalue of γ with $|\lambda_\gamma| > 1$.

Each closed geodesic c_γ determines a quadratic subfield K_γ of the quaternion algebra B. Specifically, $K_\gamma = K(\lambda_\gamma)$.

Class field theory shows that up to isomorphism, B is determined by its quadratic subfields.

This already proves Reid's theorem.

To prove our theorem we need to make this class field theory result effective and show that each subfield contributes a geodesic of bounded length.

Growth Rate of Commensurability Classes

Given a commensurability class $\mathcal{C}$ of arithmetic hyperbolic 2-manifolds, we define the volume $V_{\mathcal{C}}$ of $\mathcal{C}$ to be the minimum volume achieved by its members.

Growth Rate of Commensurability Classes

Given a commensurability class $\mathcal{C}$ of arithmetic hyperbolic 2-manifolds, we define the volume $V_{\mathcal{C}}$ of $\mathcal{C}$ to be the minimum volume achieved by its members.

Theorem (Linowitz, McReynolds, Pollack, T.)

Let k be a totally real number field of degree n_k and let $N_k(V)$ denote the number of commensurability classes $\mathcal{C}$ of compact arithmetic hyperbolic 2-manifolds arising from quaternion algebras over k with $V_{\mathcal{C}} \leq V$. Then for all sufficiently large V we have

$$N_k(V) \ll \frac{\kappa 2^{n_k - 1} V^{130}}{\zeta_k(2)},$$

where $\zeta_k(s)$ is the Dedekind zeta function of k and κ is the residue of $\zeta_k(s)$ at $s = 1$.

Counting Arithmetic Manifolds

Let $\pi(V,S)$ denote the maximum cardinality of a collection of pairwise non-commensurable arithmetic hyperbolic 2–orbifolds derived from quaternion algebras, each of which has volume less than V and geodesic length spectrum containing S.

Theorem (Linowitz, McReynolds, Pollack, T., 2014)

If $\pi(V,S) \to \infty$ as $V \to \infty$, then there are integers $1 \leq r, s \leq |S|$ and constants $c_1, c_2 > 0$ such that

$$\frac{c_1 V}{\log(V)^{1-\frac{1}{2^r}}} \leq \pi(V,S) \leq \frac{c_2 V}{\log(V)^{1-\frac{1}{2^s}}}$$

for all sufficiently large V.

Thank you!