Prime gaps

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Bounded gaps between primes: a brief history

Conjecture (de Polignac, 1849)

For even integers h, there are infinitely many pairs of primes $p, p+h$.

We refer to such values of h as *de Polignac numbers*.

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It is widely conjectured that $h = 2$ is the smallest de Polignac number, i.e., that there are infinitely many pairs of **twin primes**.

A (crude) heuristic approach

A heuristic lemma

Theorem (Chebyshev, 1852)

Approximately $\frac{1}{\log x}$ *of the integers in* $[1, x]$ *are prime.*

A crude heuristic approach

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If these two events are **independent**, then

$$\begin{aligned} P(p \text{ and } p+2 \text{ prime}) &= P(p \text{ prime}) \cdot P(p+2 \text{ prime}) \\ &= \frac{1}{\log x} \cdot \frac{1}{\log x} \\ &= \left(\frac{1}{\log x}\right)^2. \end{aligned}$$

A crude heuristic approach

Thus, we'd expect

$$\#\{p \leq x : p \text{ and } p+2 \text{ prime}\} \approx \frac{x}{(\log x)^2}.$$

Since $\lim_{x\to\infty} \frac{x}{(\log x)^2} = \infty$, this gives us reason to believe that there are infinitely many pairs of twin primes!

A problem with our crude heuristic

Problem: The events "p prime" and "$p+2$ prime" aren't independent!

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One could show (using an analogous argument) that

$$\#\{p \le x : p \text{ and } p+1 \text{ prime}\} \approx \frac{x}{(\log x)^2},$$

which is clearly false!

Refining our heuristic...

An improved heuristic

To correct for non-independence, let p and p' be independently chosen random integers. Look at:

$$\frac{P(p, p+2 \text{ not both divisible by } q)}{P(p, p' \text{ not both divisible by } q)},$$

for each small prime q.

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$$P(q \nmid p) = 1 - \frac{1}{q}.$$

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Thus, we have

$$P(q \nmid p \text{ and } q \nmid p') = \left(1 - \frac{1}{q}\right)^2.$$

An improved heuristic

$$P(q \nmid p \text{ and } q \nmid (p+2)) = P(p \not\equiv 0 \text{ or } -2 \pmod{q})$$

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Hence, if $q > 2$ then the correction factor for divisibility by q is

$$\frac{(1 - \frac{2}{q})}{(1 - \frac{1}{q})^2}.$$

If $q = 2$ then the correction factor is

$$\frac{1 - \frac{1}{2}}{(1 - \frac{1}{2})^2} = 2.$$

The heuristic with correction factor

Thus, we define

$$C := 2 \prod_{\substack{q \text{ prime} \\ q \geq 3}} \frac{(1 - 2/q)}{(1 - 1/q)^2} \approx 1.3203236\ldots.$$

This suggests that

$$\#\{p \leq x : p \text{ and } p + 2 \text{ prime}\} \approx C \frac{x}{(\log x)^2}.$$

Where the heuristics fail

The heuristic argument relies heavily on the assumption that the primes p are *uniformly distributed* among the residue classes $\pmod q$.

Let $\pi(x; q, a) := \#\{p \leq x : p \equiv a \pmod q\}$.

If the primes **were** uniformly distributed (mod q), we'd expect:

$$\pi(x; q, a) \approx \frac{x}{\varphi(q) \log x},$$

when $\gcd(a, q) = 1$.

Equidistribution for "small" q

Theorem (Bombieri-Vinogradov)

For any constant $A > 0$, *there exists* $B = B(A)$ *such that*

$$\sum_{q \le Q} \max_{\substack{a \pmod{q} \\ (a,q)=1}} \left| \pi(x; q, a) - \frac{x}{\varphi(q) \log x} \right| \ll_A \frac{x}{(\log x)^A},$$

where $Q = \frac{x^{1/2}}{(\log x)^B}$.

Equidistribution for all q?

Conjecture (Elliott-Halberstam)

The Bombieri-Vinogradov theorem still holds if we take $Q = x^\theta$, for any $\theta < 1$.

We call θ the *level of distribution* of the set of primes.

Recent results on bounded gaps between primes

Admissible k-tuples

Definition

We say that a k-tuple $(h_1, ..., h_k)$ of nonnegative integers is *admissible* if it doesn't cover all of the possible remainders (mod p) for any prime p.

Example: $(0, 2, 6, 8, 12)$ is an admissible 5-tuple.

Residue classes not covered:

$1 \pmod 2$

$1 \pmod 3$

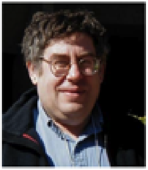
$4 \pmod 5$

$3 \pmod 7$

$3 \pmod{11}$

A conditional proof of the Bounded Gaps Theorem

G P Y



Theorem (Goldston, Pintz and Yıldırım, 2009)

If $(h_1, ..., h_k)$ is admissible and the Elliot-Halberstam Conjecture holds with $Q = x^{1/2+\eta}$, then there are infinitely many n such that at least 2 of $n + h_1, ..., n + h_k$ are prime.

Zhang's "relaxation" of Bombieri-Vinogradov

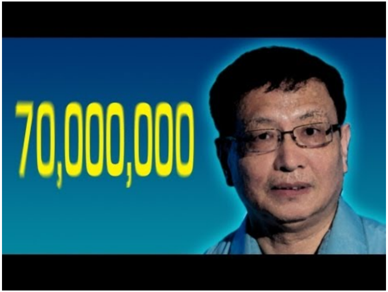
Theorem (Zhang, 2013)

There exist $\eta, \delta > 0$ *such that for any given* a,

$$\sum_{\substack{q \le Q \\ (q,a)=1 \\ q \text{ squarefree } \& \ y\text{-smooth}}} \left| \pi(x;q,a) - \frac{x}{\varphi(q)\log x} \right| \ll_A \frac{x}{(\log x)^A},$$

where $Q = x^{1/2+\eta}$ *and* $y = x^{\delta}$.

Bounded gaps between primes (at last!)



Corollary (Zhang, 2013)

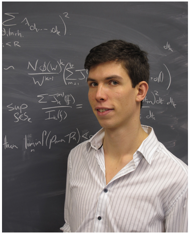
There are infinitely many pairs of primes that differ by at most $70,000,000$.

A stronger conjecture: prime k-tuples

Conjecture (Hardy-Littlewood prime k-tuples)

Let $\mathcal{H} = (h_1, \ldots, h_k)$ be admissible. Then there are infinitely many integers n such that all of $n + h_1, \ldots, n + h_k$ are prime.

Maynard and Tao's independent work



Theorem (Maynard-Tao, November 2013)

Let $m \geq 2$. There for any admissible k-tuple $\mathcal{H} = (h_1, \ldots, h_k)$ with "large enough" k (relative to m), there are infinitely many n such that at least m of $n + h_1, \ldots, n + h_k$ are prime.

The state of the art

Theorem (D. H. J. Polymath, 2014)

There are infinitely many pairs of primes that are at most 246 apart.

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Theorem (D. H. J. Polymath, 2014)

There are infinitely many pairs of primes that are at most 246 *apart.*

By assuming the Generalized Elliot–Halberstam Conjecture, this number can be reduced to 6!!!

The Maynard-Tao method

A sketch of the Maynard-Tao method

Goal: Find values of n for which the tuple $n + h_1, ..., n + h_k$ contains several primes.

Setup: For large N, look for n in the dyadic interval $[N, 2N)$.

Let $W := \prod_{p \le \log_3 N} p$. Since $\mathcal{H}$ is admissible, we can choose an integer ν so that $\gcd(\nu + h_i, W) = 1$ for all $1 \le i \le k$.

The W-trick: Pre-sieve the set to just those n satisfying $n \equiv \nu \pmod{W}$.

Thus, our sample space becomes

$$\Omega := \{N \le n < 2N : n \equiv \nu \pmod{W}\}.$$

A sketch of the Maynard-Tao method

Let $w(n)$ denote nonnegative weights and let $\chi_{\mathcal{P}}$ denote the characteristic function of the set $\mathcal{P}$ of primes. Consider

$$S_1 := \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} w(n)$$

$$S_2 := \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} \left(\sum_{i=1}^{k} \chi_{\mathcal{P}}(n + h_i) \right) w(n).$$

The fraction S_2/S_1 is a weighted average of the number of primes among $n + h_1, \ldots, n + h_k$ over Ω.

A sketch of the Maynard-Tao method

Key idea: If $S_2/S_1 > (m-1)$ for some $m \in \mathbb{Z}^+$ then at least m of $n+h_1, ..., n+h_k$ are prime, for some $n \in \Omega$.

For this method to work, one needs to select the weights $w(n)$ so that:

1. S_2 and S_1 can be estimated using tools of asymptotic analysis

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For this method to work, one needs to select the weights $w(n)$ so that:

1. S_2 and S_1 can be estimated using tools of asymptotic analysis
2. S_2/S_1 is large.

Variants of the Maynard-Tao theorem

Maynard-Tao for number fields

Theorem (Castillo, Hall, Lemke Oliver, Pollack, T., 2014)

Let $m \geq 2$. Given a number field K, there exists an integer $k_0 := k_0(m, K)$ such that for any admissible k-tuple $(h_1, ..., h_k)$ in $\mathcal{O}_K$ with $k \geq k_0$, there are infinitely many $\alpha \in \mathcal{O}_K$ such that at least m of $\alpha + h_1, ..., \alpha + h_k$ are prime.

Maynard-Tao for Chebotarev sets

A set of primes $q_1, q_2, \ldots$ has the *bounded gaps property* if $\liminf_{n\to\infty} q_{n+m} - q_n < \infty$ for every m.

Theorem (Thorner, 2014)

Chebotarev sets have the bounded gaps property.

Maynard-Tao for Chebotarev sets

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- Fix $n \in \mathbb{Z}^+$. The set of primes expressible in the form $x^2 + ny^2$.
- Let τ be the Ramanujan tau function. The set of primes p for which $\tau(p) \equiv 0 \pmod{d}$ for any positive integer d.

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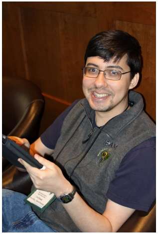
- Let τ be the Ramanujan tau function. The set of primes p for which $\tau(p) \equiv 0 \pmod{d}$ for any positive integer d.
- The set of primes p for which $\#E(\mathbb{F}_p) \equiv p + 1 \pmod{d}$ for any positive integer d.



Theorem (Hall, Ph.D. 2006; Pollack, Ph.D. 2008)

If $q \geq 3$, then any $a \in \mathbb{F}_q$ (excluding $a = 0$) occurs infinitely often as a gap between irreducible polynomials.

($q > 3$ due to Hall; $q = 3$ due to Pollack)

An improvement on Hall and Pollack's work

Theorem (Castillo, Hall, Lemke Oliver, Pollack, T., 2014)

Let $m \geq 2$. There is an integer $k_0 := k_0(m)$ such that for any admissible k-tuple $(h_1, \ldots, h_k)$ of polynomials in $\mathbb{F}_q[x]$ with $k \geq k_0$, there are infinitely many $f \in \mathbb{F}_q[x]$ such that at least m of $f + h_1, \ldots, f + h_k$ are irreducible.

Applications to runs of consecutive primes

Consecutive primes in arithmetic progressions

Theorem (Shiu, 2000)

Each set of remainders $a \pmod q$ with $\gcd(a, q) = 1$ contains arbitrarily long runs of consecutive primes.

A paradigm shift

In a paper in 2013, Banks, Freiberg and Turnage-Butterbaugh used the Maynard-Tao method to give a strikingly simple re-proof of Shiu's result.

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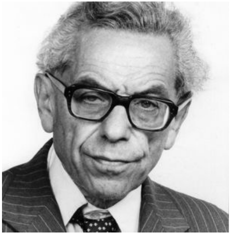
This begs the question: can the Maynard-Tao method be used to handle other "consecutive primes" problems?

A Conjecture of Erdős and Turan



For each k, let $d_k = p_{k+1} - p_k$.

Conjecture (Erdős, Turan 1948)

The sequence $\{d_k\}$ contains arbitrarily long (strictly) increasing runs and arbitrarily long (strictly) decreasing runs.

Increasing/decreasing runs of prime gaps

Theorem (Banks, Freiberg, Turnage-Butterbaugh, 2013)

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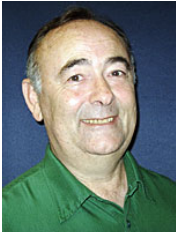
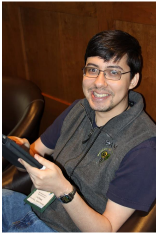
Consecutive primes with a given primitive root

Theorem (Pollack, 2014)

Under GRH, there exist arbitrarily long runs of consecutive primes possessing a given primitive root, g (where $g \neq -1$ and not a square).

Runs of primes with cyclic $E(\mathbb{F}_p)$





Theorem (Baker and Pollack, 2016)

Assume GRH. Fix an elliptic curve $E/\mathbb{Q}$. There are arbitrarily long runs of primes p for which $E(\mathbb{F}_p)$ is cyclic.

Arithmetic functions at consecutive shifted primes

Theorem (Pollack, T., 2015)

Let $f = \varphi, \sigma, \omega, \Omega, \tau$. There are arbitrarily long runs of consecutive primes p on which $f(p-1)$ is decreasing. The same holds for $f(p-1)$ increasing.

Proof sketch for decreasing runs of $\varphi(p-1)$'s

Same basic setup as in Maynard-Tao:

For large N, let $W := \prod_{p \le \log_3 N} p$ and define

$$S_1 := \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} w(n)$$

$$S_2 := \sum_{\substack{N \le n < 2N \\ n \equiv \nu \pmod{W}}} \left(\sum_{i=1}^{k} \chi_{\mathcal{P}}(n + h_i) \right) w(n).$$

We want $\frac{S_2}{S_1} > m - 1$ so that at least m of $n + h_1, \ldots, n + h_k$ are prime for $n \in \Omega$.

Proof sketch for decreasing runs of $\varphi(p-1)$'s

Key innovation: **Modify the W-trick**

Fix $\mathcal{H} = \{h_1, ..., h_k\}$ where $h_i = (i-1)(2k)!$. With this $\mathcal{H}$, we want to choose $\nu \pmod W$ such that:

1. (Consecutive Primes) For $n \in \Omega$, any prime $[n+h_1, n+h_k]$ is in $n+\mathcal{H}$.

2. (Decreasing φ values) With probability $1+o(1)$,

$$\frac{n+h_i-1}{\varphi(n+h_i-1)} \in (2^{4i}, 2^{4i+3}]$$

for $1 \le i \le k$.

Variations

Theorem (Pollack, T., 2015)

Let $f = \varphi, \sigma, \omega, \Omega, \tau$. There are arbitrarily long runs of consecutive primes p on which $f(p-1)$ is decreasing. The same holds for $f(p-1)$ increasing.

Variation	Modification
Increasing φ's	Replace $\mathcal{H}$ with $-\mathcal{H}$
Increasing σ's	Replace $\frac{n+h_i-1}{\varphi(n+h_i-1)}$ with $\frac{\sigma(n+h_i-1)}{n+h_i-1}$.
Decreasing σ's	Like increasing σ's but replace $\mathcal{H}$ with $-\mathcal{H}$.

**Need additional modifications to handle ω, Ω, τ.

Digit sums of consecutive primes

Definition

Let $s_g(n)$ denote the sum of the base-g digits of n.

Example: $s_{10}(523) = 5 + 2 + 3 = 10$

$$s_{10}(541) = 5 + 4 + 1 = 10$$

Question (Sierpinski, 1961): Are there arbitrarily long runs of consecutive primes p on which $s_g(p)$ is constant? increasing? decreasing?

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A brief history:

- Sierpinski (1961): $s_{10}(p_n) < s_{10}(p_{n+1})$ infinitely often.
- Erdős (1962): $s_{10}(p_n) > s_{10}(p_{n+1})$ infinitely often.
- Sierpinski (1968): **Assuming Dickson's prime k-tuples conjecture**, $s_{10}(p_n) > s_{10}(p_{n+1}) > s_{10}(p_{n+2})$ infinitely often.

Digit sums of consecutive primes

A brief history:

- Sierpinski (1961): $s_{10}(p_n) < s_{10}(p_{n+1})$ infinitely often.
- Erdős (1962): $s_{10}(p_n) > s_{10}(p_{n+1})$ infinitely often.
- Sierpinski (1968): **Assuming Dickson's prime k-tuples conjecture**, $s_{10}(p_n) > s_{10}(p_{n+1}) > s_{10}(p_{n+2})$ infinitely often.
- Schinzel (unpublished claim): **Assuming Hypothesis H**, there are arbitrarily long runs of consecutive p on which $s_{10}(p)$ is increasing (decreasing).

Digit sums of consecutive primes

Theorem (Pollack, T., 2015)

For any base g, there are arbitrarily long runs of consecutive primes p on which $s_g(p)$ is constant/increasing/decreasing.

Thank you!