Bertrand's Postulate

Lola Thompson

Ross Program

July 3, 2009

Bertrand's Postulate

"I've said it once and I'll say it again: There's always a prime between n and $2n$."

-Joseph Bertrand, conjectured in 1845

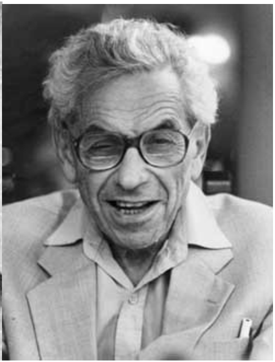
Bertrand's Postulate

Bertrand did not prove his postulate. He verified the statement (by hand) for all positive integers n up to 6,000,000.

In 1850, Chebyshev proved Bertrand's Postulate. For this reason, it is also referred to as "Chebyshev's Theorem."

Bertrand's Postulate

Many other proofs have been found in the time since Chebyshev first proved this theorem. We will follow a proof due to Ramanujan and Erdös.

Outline

Fun With Binomial Coefficients

Definition

$n! = \#$ of ways of ordering n elements

Definition

$\binom{n}{k} = \#$ of ways of choosing k elements from a set containing n objects, without worrying about order.

Fun With Binomial Coefficients

- There is a curious relationship between $\binom{n}{k}$ and Pascal's Triangle.

Fun With Binomial Coefficients

- There is a curious relationship between $\binom{n}{k}$ and Pascal's Triangle.
- If we write down Pascal's Triangle, the first few rows are:
  1
  1 1
  1 2 1
  1 3 3 1

Fun With Binomial Coefficients

- There is a curious relationship between $\binom{n}{k}$ and Pascal's Triangle.
- If we write down Pascal's Triangle, the first few rows are:
  1
  1 1
  1 2 1
  1 3 3 1
- We can obtain the numbers in the next row by adding adjacent pairs of numbers from the previous row:
  1 3 3 1
  1 4 6 4 1

Fun With Binomial Coefficients

- There is a curious relationship between $\binom{n}{k}$ and Pascal's Triangle.
- If we write down Pascal's Triangle, the first few rows are:
  1
  1 1
  1 2 1
  1 3 3 1
- We can obtain the numbers in the next row by adding adjacent pairs of numbers from the previous row:
  1 3 3 1
  1 4 6 4 1
- The elements in the 4^{th} row are precisely $\binom{4}{0}$ through $\binom{4}{4}$.

Fun With Binomial Coefficients

- Another interesting observation that you might make is that, at least in the examples above, the sum of all of the elements in a row is a power of 2.

Fun With Binomial Coefficients

- Another interesting observation that you might make is that, at least in the examples above, the sum of all of the elements in a row is a power of 2.
- For example, $1 + 2 + 1 = 2^2$ and $1 + 3 + 3 + 1 = 2^3$.

Fun With Binomial Coefficients

- Another interesting observation that you might make is that, at least in the examples above, the sum of all of the elements in a row is a power of 2.
- For example, $1 + 2 + 1 = 2^2$ and $1 + 3 + 3 + 1 = 2^3$.
- **Exercise**: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Fun With Binomial Coefficients

- One particular type of binomial coefficient will be of interest to us as we prove Bertrand's Postulate. That coefficient is $\binom{2n}{n}$, the coefficient in the center of the $2n^{th}$ row of Pascal's Triangle.

Fun With Binomial Coefficients

- One particular type of binomial coefficient will be of interest to us as we prove Bertrand's Postulate. That coefficient is $\binom{2n}{n}$, the coefficient in the center of the $2n^{th}$ row of Pascal's Triangle.
- Which primes divide $\binom{2n}{n}$? Let's look at some examples.

Fun With Binomial Coefficients

- One particular type of binomial coefficient will be of interest to us as we prove Bertrand's Postulate. That coefficient is $\binom{2n}{n}$, the coefficient in the center of the $2n^{th}$ row of Pascal's Triangle.
- Which primes divide $\binom{2n}{n}$? Let's look at some examples.
- $\binom{4}{2} = 6$. Which primes divide 6? What about $\binom{6}{3}$? $\binom{8}{4}$? $\binom{10}{5}$? Any conjectures about when a prime has to divide $\binom{2n}{n}$?

Fun With Binomial Coefficients

- One particular type of binomial coefficient will be of interest to us as we prove Bertrand's Postulate. That coefficient is $\binom{2n}{n}$, the coefficient in the center of the $2n^{th}$ row of Pascal's Triangle.
- Which primes divide $\binom{2n}{n}$? Let's look at some examples.
- $\binom{4}{2} = 6$. Which primes divide 6? What about $\binom{6}{3}$? $\binom{8}{4}$? $\binom{10}{5}$? Any conjectures about when a prime has to divide $\binom{2n}{n}$?
- **Exercise**: $p \mid \binom{2n}{n}$ if there exists a positive integer j with $n < p^j \leq 2n$

Fun With Binomial Coefficients

Assuming that those exercises are true, we can prove two results:

Lemma (1)

$$\prod_{p:n<p^j\leq 2n} p \mid \binom{2n}{n}.$$

Fun With Binomial Coefficients

Assuming that those exercises are true, we can prove two results:

Lemma (1)

$$\prod_{p: n < p^j \leq 2n} p \mid \binom{2n}{n}.$$

Proof We know that each prime p with $n < p^j \leq 2n$ divides $\binom{2n}{n}$.Since the primes are distinct, they are pairwise relatively prime.Thus, their product must divide $\binom{2n}{n}$.

Fun With Binomial Coefficients

Assuming that those exercises are true, we can prove two results:

Lemma (1)

$$\prod_{p:n<p^j\leq 2n} p \mid \binom{2n}{n}.$$

Proof We know that each prime p with $n < p^j \leq 2n$ divides $\binom{2n}{n}$. Since the primes are distinct, they are pairwise relatively prime. Thus, their product must divide $\binom{2n}{n}$.

Lemma (2)

$\binom{2n}{n} \leq 4^n$ *for all* $n \geq 0$.

Fun With Binomial Coefficients

Assuming that those exercises are true, we can prove two results:

Lemma (1)

$$\prod_{p:n<p^j\leq 2n} p \mid \binom{2n}{n}.$$

Proof We know that each prime p with $n < p^j \leq 2n$ divides $\binom{2n}{n}$.Since the primes are distinct, they are pairwise relatively prime.Thus, their product must divide $\binom{2n}{n}$.

Lemma (2)

$\binom{2n}{n} \leq 4^n$ *for all* $n \geq 0$.

Proof Look at the $2n^{th}$ row of Pascal's Triangle: $\binom{2n}{n}$ is in the center, $2^{2n} =$ sum of all terms in the row.

A Few New Arithmetic Functions

We're already familiar with σ, τ and φ. Let's define some new arithmetic functions:

Definition

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, k > 0 \\ 0, & \textit{otherwise} \end{cases}$$

A Few New Arithmetic Functions

We're already familiar with σ, τ and φ. Let's define some new arithmetic functions:

Definition

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, k > 0 \\ 0, & \textit{otherwise} \end{cases}$$

Examples: $\Lambda(2) = \log 2$, $\Lambda(4) = \log 2$, $\Lambda(6) = 0$.

A Few New Arithmetic Functions

We're already familiar with σ, τ and φ. Let's define some new arithmetic functions:

Definition

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, k > 0 \\ 0, & \textit{otherwise} \end{cases}$$

Examples: $\Lambda(2) = \log 2$, $\Lambda(4) = \log 2$, $\Lambda(6) = 0$.

Definition

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

A Few New Arithmetic Functions

We're already familiar with σ, τ and φ. Let's define some new arithmetic functions:

Definition

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, k > 0 \\ 0, & \textit{otherwise} \end{cases}$$

Examples: $\Lambda(2) = \log 2$, $\Lambda(4) = \log 2$, $\Lambda(6) = 0$.

Definition

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

Example:
$\psi(6) = 0 + \log 2 + \log 3 + \log 2 + \log 5 + 0 = \log(2^2 \cdot 3 \cdot 5) = \log(60)$.

An Upper Bound for $\psi(x)$

Lemma

$\psi(x) \leq x$ *log 4*

For now, let's pretend that $x \in \mathbb{Z}$. This will allow us to use the Well-Ordering Principle.

Base Case: If $n = 1$, then $\psi(1) = \sum_{n \leq 1} \Lambda(n) = \Lambda(1) = 0$.

(This is certainly $\leq 1 \cdot \log 4$)

Proof of Upper Bound for $\psi(x)$

Let $S = \{x \in \mathbb{Z}^+ \mid \psi(x) > x \log 4\}$. Assume that S is non-empty. Then, by Well-Ordering Principle, S has a least element. Call it l.

Case 1: $l = 2k$

- $\psi(2k) - \psi(k)$

Proof of Upper Bound for $\psi(x)$

Let $S = \{x \in \mathbb{Z}^+ \mid \psi(x) > x \log 4\}$. Assume that S is non-empty. Then, by Well-Ordering Principle, S has a least element. Call it l.

Case 1: $l = 2k$

- $\psi(2k) - \psi(k)$
- $= \sum_{n \leq 2k} \Lambda(n) - \sum_{n \leq k} \Lambda(n)$ (from definition of ψ)

Proof of Upper Bound for $\psi(x)$

Let $S = \{x \in \mathbb{Z}^+ \mid \psi(x) > x \log 4\}$. Assume that S is non-empty. Then, by Well-Ordering Principle, S has a least element. Call it l.

Case 1: $l = 2k$

- $\psi(2k) - \psi(k)$
- $= \sum_{n \leq 2k} \Lambda(n) - \sum_{n \leq k} \Lambda(n)$ (from definition of ψ)
- $= \sum_{k < n \leq 2k} \Lambda(n)$ (canceling terms)

Proof of Upper Bound for $\psi(x)$

Let $S = \{x \in \mathbb{Z}^+ \mid \psi(x) > x \log 4\}$. Assume that S is non-empty. Then, by Well-Ordering Principle, S has a least element. Call it l.

Case 1: $l = 2k$

- $\psi(2k) - \psi(k)$
- $= \sum_{n \leq 2k} \Lambda(n) - \sum_{n \leq k} \Lambda(n)$ (from definition of ψ)
- $= \sum_{k < n \leq 2k} \Lambda(n)$ (canceling terms)
- $\leq \log \binom{2k}{k}$ (since $\Lambda(n) = p$ if $n = p^j$ and 0 otherwise, so this line follows from Lemma (1) after taking log of both sides)

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.
- From Lemma (2), $\binom{2k}{k} \leq 4^k$, i.e. $\psi(2k) - \psi(k) \leq \log(4^k)$.

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.
- From Lemma (2), $\binom{2k}{k} \leq 4^k$, i.e. $\psi(2k) - \psi(k) \leq \log(4^k)$.
- From high school math: $\log(4^k) = k \log 4$.

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.
- From Lemma (2), $\binom{2k}{k} \leq 4^k$, i.e. $\psi(2k) - \psi(k) \leq \log(4^k)$.
- From high school math: $\log(4^k) = k \log 4$.
- So $\psi(2k) \leq \psi(k) + k \log 4$

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.
- From Lemma (2), $\binom{2k}{k} \leq 4^k$, i.e. $\psi(2k) - \psi(k) \leq \log(4^k)$.
- From high school math: $\log(4^k) = k \log 4$.
- So $\psi(2k) \leq \psi(k) + k \log 4$
- $\leq k \log 4 + k \log 4$ (since $l = 2k$ was the smallest element in S, so $k \notin S \Rightarrow \psi(k) \leq k \log 4$)

Proof of Upper Bound for $\psi(x)$

- From the previous slide: $\psi(2k) - \psi(k) \leq \log \binom{2k}{k}$.
- From Lemma (2), $\binom{2k}{k} \leq 4^k$, i.e. $\psi(2k) - \psi(k) \leq \log(4^k)$.
- From high school math: $\log(4^k) = k \log 4$.
- So $\psi(2k) \leq \psi(k) + k \log 4$
- $\leq k \log 4 + k \log 4$ (since $l = 2k$ was the smallest element in S, so $k \notin S \Rightarrow \psi(k) \leq k \log 4$)
- $\therefore \psi(2k) \leq 2k \log 4$, ie. l cannot be even.

Proof of Upper Bound for $\psi(x)$

Case 2: $l = 2k + 1$.

The argument for the case where l is odd is similar to the case where l is even. The result is the same, i.e. it shows that l cannot be odd.
Since the least element of S is neither even nor odd then S must empty.

- Remark: The inequality $\psi(x) \leq x \log 4$ holds for ALL $x \geq 1$ (not just integers). Why?

Proof of Upper Bound for $\psi(x)$

Case 2: $l = 2k + 1$.

The argument for the case where l is odd is similar to the case where l is even. The result is the same, i.e. it shows that l cannot be odd. Since the least element of S is neither even nor odd then S must empty.

- Remark: The inequality $\psi(x) \leq x \log 4$ holds for ALL $x \geq 1$ (not just integers). Why?
- $\psi(x) = \psi(\lfloor x \rfloor) \leq \lfloor x \rfloor \log 4 \leq x \log 4$.

Proving Bertrand's Postulate

We will use what we have learned about $\binom{2n}{n}$ and $\psi(x)$ in order to prove our main result:

Theorem (Bertrand's Postulate)

For every $n \in \mathbb{Z}^+$, there exists a prime p such that $n < p \leq 2n$.

The Setup : $ABC = \binom{2n}{n}$

- Let $A = \prod_{n<p\leq 2n} p$.

The Setup : $ABC = \binom{2n}{n}$

- Let $A = \prod_{n<p\leq 2n} p$.

- From Lemma (1), we know that $A \mid \binom{2n}{n}$. So, there exists $m \in \mathbb{Z}$ st. $A \cdot m = \binom{2n}{n}$.

The Setup : $ABC = \binom{2n}{n}$

- Let $A = \prod_{n<p\leq 2n} p$.
- From Lemma (1), we know that $A \mid \binom{2n}{n}$. So, there exists $m \in \mathbb{Z}$ st. $A \cdot m = \binom{2n}{n}$.
- By the Unique Factorization Theorem, we can factor m into a product of primes (uniquely).

The Setup : $ABC = \binom{2n}{n}$

- Let $A = \prod_{n<p\leq 2n} p$.
- From Lemma (1), we know that $A \mid \binom{2n}{n}$. So, there exists $m \in \mathbb{Z}$ st. $A \cdot m = \binom{2n}{n}$.
- By the Unique Factorization Theorem, we can factor m into a product of primes (uniquely).
- Let $B =$ contribution to $\binom{2n}{n}$ from primes $p \in (\sqrt{2n}, n]$.
  Let $C =$ contribution to $\binom{2n}{n}$ from primes $p \leq \sqrt{2n}$.

The Setup : $ABC = \binom{2n}{n}$

- Let $A = \prod_{n<p\leq 2n} p$.
- From Lemma (1), we know that $A \mid \binom{2n}{n}$. So, there exists $m \in \mathbb{Z}$ st. $A \cdot m = \binom{2n}{n}$.
- By the Unique Factorization Theorem, we can factor m into a product of primes (uniquely).
- Let $B =$ contribution to $\binom{2n}{n}$ from primes $p \in (\sqrt{2n}, n]$.
  Let $C =$ contribution to $\binom{2n}{n}$ from primes $p \leq \sqrt{2n}$.
- Then $ABC = \binom{2n}{n}$.

The Setup : $ABC = \binom{2n}{n}$

Goal: We want to show that $BC < \binom{2n}{n}$.

Why does this imply that Bertrand's Postulate holds?

- If $BC < \binom{2n}{n}$ then $A \neq 1$.

The Setup : $ABC = \binom{2n}{n}$

Goal: We want to show that $BC < \binom{2n}{n}$.

Why does this imply that Bertrand's Postulate holds?

- If $BC < \binom{2n}{n}$ then $A \neq 1$.
- But $A = \prod_{n \leq p \leq 2n} p$, so there must be a prime dividing A.

The Setup : $ABC = \binom{2n}{n}$

Goal: We want to show that $BC < \binom{2n}{n}$.

Why does this imply that Bertrand's Postulate holds?

- If $BC < \binom{2n}{n}$ then $A \neq 1$.
- But $A = \prod_{n \leq p \leq 2n} p$, so there must be a prime dividing A.
- In other words, there must be a prime between n and $2n$ dividing $\binom{2n}{n}$.

The Setup : $ABC = \binom{2n}{n}$

Goal: We want to show that $BC < \binom{2n}{n}$.

Why does this imply that Bertrand's Postulate holds?

- If $BC < \binom{2n}{n}$ then $A \neq 1$.
- But $A = \prod_{n \leq p \leq 2n} p$, so there must be a prime dividing A.
- In other words, there must be a prime between n and $2n$ dividing $\binom{2n}{n}$.
- This proves Bertrand's Postulate because it proves the existence of a prime between n and $2n$.

The Setup : $ABC = \binom{2n}{n}$

In order to show that $BC < \binom{2n}{n}$, we will find upper bounds for B and C and we will find a lower bound for $\binom{2n}{n}$.

A Lower Bound for $\binom{2n}{n}$

Lemma

$\binom{2n}{n} \geq \frac{4^n}{2n}$

Proof

- In an earlier exercise, we showed that $\displaystyle\sum_{j=0}^{2n} \binom{2n}{j} = 4^n$.

A Lower Bound for $\binom{2n}{n}$

Lemma

$\binom{2n}{n} \geq \frac{4^n}{2n}$

Proof

- In an earlier exercise, we showed that $\displaystyle\sum_{j=0}^{2n} \binom{2n}{j} = 4^n$.
- Since the two end terms in a row of Pascal's triangle are both 1, then $\displaystyle\sum_{j=0}^{2n} \binom{2n}{j} = 2 + \sum_{j=1}^{2n-1} \binom{2n}{j}$.

A Lower Bound for $\binom{2n}{n}$

Lemma

$\binom{2n}{n} \geq \frac{4^n}{2n}$

Proof

- In an earlier exercise, we showed that $\sum_{j=0}^{2n} \binom{2n}{j} = 4^n$.
- Since the two end terms in a row of Pascal's triangle are both 1, then $\sum_{j=0}^{2n} \binom{2n}{j} = 2 + \sum_{j=1}^{2n-1} \binom{2n}{j}$.
- We know that the middle term, $\binom{2n}{n}$, is the largest term in the sum.

A Lower Bound for $\binom{2n}{n}$

Lemma

$\binom{2n}{n} \geq \frac{4^n}{2n}$

Proof

- In an earlier exercise, we showed that $\sum_{j=0}^{2n} \binom{2n}{j} = 4^n$.
- Since the two end terms in a row of Pascal's triangle are both 1, then $\sum_{j=0}^{2n} \binom{2n}{j} = 2 + \sum_{j=1}^{2n-1} \binom{2n}{j}$.
- We know that the middle term, $\binom{2n}{n}$, is the largest term in the sum.
- Thus, $2 + \sum_{j=1}^{2n-1} \binom{2n}{j} \leq (2n)\binom{2n}{n}$ since we are summing $2n$ terms.

Another Pair of Useful Exercises

The following exercises will also be useful in bounding BC:

Exercise Prove or disprove and salvage if possible: If $x \in \mathbb{R}$, then $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 0$ or 1.

Exercise (a) What power of the prime p appears in the prime factorization of $n!$?
(b) What power of p appears in the factorization of $\binom{n}{k}$?

An Upper Bound for C

Lemma

$C \leq (2n)^{\sqrt{2n}-1}$

Proof

- Let k be the highest power of p dividing $\binom{2n}{n}$. Have you solved both of the exercises on the previous slide? Once you have, you will see that $k \leq \sum_{j : p^j \leq 2n} 1$.

An Upper Bound for C

Lemma

$C \leq (2n)^{\sqrt{2n}-1}$

Proof

- Let k be the highest power of p dividing $\binom{2n}{n}$. Have you solved both of the exercises on the previous slide? Once you have, you will see that $k \leq \sum_{j:p^j \leq 2n} 1$.
- The sum on the right counts the number of j that satisfy $p^j \leq 2n$.

An Upper Bound for C

Lemma

$C \leq (2n)^{\sqrt{2n}-1}$

Proof

- Let k be the highest power of p dividing $\binom{2n}{n}$. Have you solved both of the exercises on the previous slide? Once you have, you will see that $k \leq \sum_{j : p^j \leq 2n} 1$.
- The sum on the right counts the number of j that satisfy $p^j \leq 2n$.
- In order to determine that number, we need to solve the equation $p^x \leq 2n$. But this is the same as solving $x \log p \leq \log 2n$.

An Upper Bound for C

Lemma

$C \leq (2n)^{\sqrt{2n}-1}$

Proof

- Let k be the highest power of p dividing $\binom{2n}{n}$. Have you solved both of the exercises on the previous slide? Once you have, you will see that $k \leq \sum_{j: p^j \leq 2n} 1$.
- The sum on the right counts the number of j that satisfy $p^j \leq 2n$.
- In order to determine that number, we need to solve the equation $p^x \leq 2n$. But this is the same as solving $x \log p \leq \log 2n$.
- Thus, $x \leq \frac{\log 2n}{\log p}$.

An Upper Bound for C

Since the j that satisfy $p^j \leq 2n$ must be integers,

then $\displaystyle\sum_{j:p^j \leq 2n} 1 = \lfloor \frac{\log 2n}{\log p} \rfloor.$

Recall that $C = \displaystyle\prod_{\substack{p \leq \sqrt{2n} \\ p \mid \binom{2n}{n}}} p.$

Suppose that a prime p is not included in C, i.e. $p > \sqrt{2n}$ and $p \leq 2n$.

Then $\lfloor \frac{\log 2n}{\log p} \rfloor = 1$.

So all of the primes p such that $p^k \mid \binom{2n}{n}$ for $k > 1$ must occur in C.

An Upper Bound for C

Recalling that $C = \prod_{\substack{p \le \sqrt{2n} \\ p \mid \binom{2n}{n}}} p$, we have

$$C \le \prod_{p \le \sqrt{2n}} p^{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor}$$

- $$\le \prod_{p \le \sqrt{2n}} p^{\frac{\log 2n}{\log p}}$$

An Upper Bound for C

Recalling that $C = \prod_{\substack{p \leq \sqrt{2n} \\ p | \binom{2n}{n}}} p$, we have

$$C \leq \prod_{p \leq \sqrt{2n}} p^{\lfloor \frac{\log 2n}{\log p} \rfloor}$$

- $\leq \prod_{p \leq \sqrt{2n}} p^{\frac{\log 2n}{\log p}}$
- $= \prod_{p \leq \sqrt{2n}} e^{\log 2n}$ (log rule: $p^{\alpha} = e^{\alpha \log p}$)

An Upper Bound for C

Recalling that $C = \prod_{\substack{p \leq \sqrt{2n} \\ p \mid \binom{2n}{n}}} p$, we have

$$C \leq \prod_{p \leq \sqrt{2n}} p^{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor}$$

- $$\leq \prod_{p \leq \sqrt{2n}} p^{\frac{\log 2n}{\log p}}$$
- $$= \prod_{p \leq \sqrt{2n}} e^{\log 2n} \text{ (log rule: } p^{\alpha} = e^{\alpha \log p}\text{)}$$
- $$= \prod_{p \leq \sqrt{2n}} 2n$$

An Upper Bound for C

- But $\prod_{p \leq \sqrt{2n}} 2n = (2n)^{\pi(\sqrt{2n})}$, where $\pi(\sqrt{2n}) = \#$ of primes $\leq \sqrt{2n}$.

An Upper Bound for C

- But $\prod_{p \le \sqrt{2n}} 2n = (2n)^{\pi(\sqrt{2n})}$, where $\pi(\sqrt{2n}) = \#$ of primes $\le \sqrt{2n}$.
- Since 1 is not prime, then $\pi(\sqrt{2n}) \le \sqrt{2n} - 1$.

An Upper Bound for C

- But $\prod_{p \leq \sqrt{2n}} 2n = (2n)^{\pi(\sqrt{2n})}$, where $\pi(\sqrt{2n}) = \#$ of primes $\leq \sqrt{2n}$.
- Since 1 is not prime, then $\pi(\sqrt{2n}) \leq \sqrt{2n} - 1$.
- Thus, we see that $C \leq (2n)^{\sqrt{2n}-1}$.

An Upper Bound for B

Lemma

$B \leq 4^{\frac{2}{3}n}$

Proof

- Recall that $B = \prod_{\substack{\sqrt{2n} < p \leq n \\ p \mid \binom{2n}{n}}} p$.

An Upper Bound for B

Lemma

$B \leq 4^{\frac{2}{3}n}$

Proof

- Recall that $B = \prod_{\substack{\sqrt{2n} < p \leq n \\ p \mid \binom{2n}{n}}} p.$
- For all $n > 4.5$, $\sqrt{2n} < \frac{2}{3}n$
  (square both sides, then divide both sides by n)

An Upper Bound for B

Lemma

$B \leq 4^{\frac{2}{3}n}$

Proof

- Recall that $B = \prod_{\substack{\sqrt{2n} < p \leq n \\ p \mid \binom{2n}{n}}} p$.
- For all $n > 4.5$, $\sqrt{2n} < \frac{2}{3}n$
  (square both sides, then divide both sides by n)
- We will separate B into two products: $\prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p$ and $\prod_{\frac{2}{3}n < p \leq n} p$.

An Upper Bound for B

- Let $p \in (\frac{2}{3}n, n]$.

An Upper Bound for B

- Let $p \in (\frac{2}{3}n, n]$.
- We will show that $k = 0$ when p is in this range, where k is the highest power of p dividing $\binom{2n}{n}$.

An Upper Bound for B

- Let $p \in (\frac{2}{3}n, n]$.
- We will show that $k = 0$ when p is in this range, where k is the highest power of p dividing $\binom{2n}{n}$.
- Since $p \in (\frac{2}{3}n, n]$, we have $1 \leq \frac{n}{p} < \frac{3}{2}$.
  Thus $\frac{n}{p} = 1 + r$, with $0 \leq r < \frac{1}{2}$.

An Upper Bound for B

- Let $p \in (\frac{2}{3}n, n]$.
- We will show that $k = 0$ when p is in this range, where k is the highest power of p dividing $\binom{2n}{n}$.
- Since $p \in (\frac{2}{3}n, n]$, we have $1 \leq \frac{n}{p} < \frac{3}{2}$.
Thus $\frac{n}{p} = 1 + r$, with $0 \leq r < \frac{1}{2}$.
- Using this fact with the formula for k that you found in the problem set yields $k = 0$. So, the product $\prod_{\frac{2}{3}n<p\leq n} p$ doesn't contribute any primes to B.

An Upper Bound for B

- Let $p \in (\frac{2}{3}n, n]$.
- We will show that $k = 0$ when p is in this range, where k is the highest power of p dividing $\binom{2n}{n}$.
- Since $p \in (\frac{2}{3}n, n]$, we have $1 \leq \frac{n}{p} < \frac{3}{2}$. Thus $\frac{n}{p} = 1 + r$, with $0 \leq r < \frac{1}{2}$.
- Using this fact with the formula for k that you found in the problem set yields $k = 0$. So, the product $\displaystyle\prod_{\frac{2}{3}n < p \leq n} p$ doesn't contribute any primes to B.
- $\therefore B = \displaystyle\prod_{\substack{\sqrt{2n} < p \leq n \\ p \mid \binom{2n}{n}}} p \leq \prod_{p \leq \frac{2}{3}n} p \leq 4^{\frac{2}{3}n}$ (since $\psi(x) \leq x \log 4$).

Putting Everything Together

What we have shown:

$\binom{2n}{n} = ABC$

$\binom{2n}{n} \geq \frac{4^n}{2n}$

$C \leq (2n)^{\sqrt{2n}-1}$

$B \leq 4^{\frac{2}{3}n}$

- So, $A = \binom{2n}{n}/(BC)$

Putting Everything Together

What we have shown:

$\binom{2n}{n} = ABC$

$\binom{2n}{n} \geq \frac{4^n}{2n}$

$C \leq (2n)^{\sqrt{2n}-1}$

$B \leq 4^{\frac{2}{3}n}$

- So, $A = \binom{2n}{n}/(BC)$
- $\geq \frac{\frac{4^n}{2n}}{(2n)^{\sqrt{2n}-1} . 4^{\frac{2}{3}n}}$

Putting Everything Together

What we have shown:

$\binom{2n}{n} = ABC$

$\binom{2n}{n} \geq \frac{4^n}{2n}$

$C \leq (2n)^{\sqrt{2n}-1}$

$B \leq 4^{\frac{2}{3}n}$

- So, $A = \binom{2n}{n}/(BC)$
- $\geq \frac{\frac{4^n}{2n}}{(2n)^{\sqrt{2n}-1} . 4^{\frac{2}{3}n}}$
- $= \frac{4^{\frac{1}{3}n}}{(2n)^{\sqrt{2n}}}$

Putting Everything Together

- Remember that in order for Bertrand's Postulate to hold, we need to show that $A > 1$. Thus, we need to determine when $4^{\frac{1}{3}n} > (2n)^{\sqrt{2n}}$.

Putting Everything Together

- Remember that in order for Bertrand's Postulate to hold, we need to show that $A > 1$. Thus, we need to determine when $4^{\frac{1}{3}n} > (2n)^{\sqrt{2n}}$.
- Using high school math, we can show that $4^{\frac{1}{3}n} > (2n)^{\sqrt{2n}}$ holds when $n > 450$.

Putting Everything Together

- Remember that in order for Bertrand's Postulate to hold, we need to show that $A > 1$. Thus, we need to determine when $4^{\frac{1}{3}n} > (2n)^{\sqrt{2n}}$.
- Using high school math, we can show that $4^{\frac{1}{3}n} > (2n)^{\sqrt{2n}}$ holds when $n > 450$.
- $\therefore$ Bertrand's Postulate holds for all $n > 450$.

Finishing Up

In order to conclude that Bertrand's Postulate is true for all $n \in \mathbb{Z}^+$, we just need to check values of $n \leq 450$.

Remember that Bertrand checked all n up to 6,000,000, so if you believe him then we're done!

If not, consider the list of primes $2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631$, where each is less than twice the preceding. This proves Bertrand's Postulate for all $n < 631$, since any such n can be squeezed between two numbers on the list.

Generalizations

One way that a mathematician finds new problems to solve is by looking at a result that has already been proven and asking "Does this hold in a more general setting?"

Of course, "more general" can mean many different things. For example, we showed that $\binom{2n}{n} \leq 4^n$ for all $n \geq 0$. Perhaps we could have shown a similar result for *any* integer n. Another generalization would be to try to bound $\binom{kn}{n}$, $k \in \mathbb{Z}^+$.

Can you think of a "more general" statement of Bertrand's Postulate?

Generalizations

Here are a few well-known generalizations of Bertrand's Postulate:

Theorem (Sylvester)

The product of k consecutive integers greater than k is divisible by a prime greater than k.

Generalizations

Here are a few well-known generalizations of Bertrand's Postulate:

Theorem (Sylvester)

The product of k consecutive integers greater than k is divisible by a prime greater than k.

Theorem (Erdös)

For any positive integer k, there is a natural number N such that for all $n > N$, there are at least k primes between n and $2n$.

Generalizations

Here are a few well-known generalizations of Bertrand's Postulate:

Theorem (Sylvester)

The product of k consecutive integers greater than k is divisible by a prime greater than k.

Theorem (Erdös)

For any positive integer k, there is a natural number N such that for all $n > N$, there are at least k primes between n and $2n$.

Conjecture (Legendre) For every $n > 1$, there is a prime p such that $n^2 < p < (n+1)^2$.

Some Neat Applications

Using Bertrand's Postulate, we can also prove many other interesting results, including:

- Every integer $n > 6$ can be written as a sum of distinct primes.

Some Neat Applications

Using Bertrand's Postulate, we can also prove many other interesting results, including:

- Every integer $n > 6$ can be written as a sum of distinct primes.
- $\forall N \in \mathbb{N}$, there exists an even integer $k > 0$ for which there are at least N prime pairs $p, p + k$.