

ABELIAN SURFACES OVER FINITE FIELDS WITH PRESCRIBED GROUPS

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ABSTRACT. Let A be an abelian surface over \mathbb{F}_q , the field of q elements. The rational points on A/\mathbb{F}_q form an abelian group $A(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_1n_2\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3n_4\mathbb{Z}$. We are interested in knowing which groups of this shape actually arise as the group of points on some abelian surface over some finite field. For a fixed prime power q , a characterization of the abelian groups that occur was recently found by Rybakov. One can use this characterization to obtain a set of congruences on certain combinations of coefficients of the corresponding Weil polynomials. We use Rybakov’s criterion to show that groups $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_1n_2\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3n_4\mathbb{Z}$ do not occur if n_1 is very large with respect to n_2, n_2, n_4 (Theorem 1.1), and occur with density zero in a wider range of the variables (Theorem 1.2).

1. INTRODUCTION

Let E be an elliptic curve over the finite field \mathbb{F}_p . It is well-known that the points on E over \mathbb{F}_p form a finite abelian group $E(\mathbb{F}_p)$ of “rank” at most 2, i.e.,

$$E(\mathbb{F}_p) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_1n_2\mathbb{Z}, \tag{1.1}$$

for some positive integers n_1, n_2 . It is natural to ask which groups arise in this manner as p runs through all primes and as E runs through all curves over \mathbb{F}_p . Let $S(N_1, N_2)$ be the set of pairs of integers $n_1 \leq N_1, n_2 \leq N_2$ such that there exists a prime p and a curve E/\mathbb{F}_p for which (1.1) holds. The problem of estimating the size of $S(N_1, N_2)$ was first considered by Banks, Pappalardi, and Shparlinski in [1], who gave precise conjectures and numerical evidence for this problem. In particular, they conjectured that the “very split” groups (when n_1 is very large compared to n_2) occur with density zero. This was proven by Chandee, David, Koukoulopoulos, and Smith in [2], who showed that

$$\#S(N_1, N_2) = o(N_1N_2)$$

when $N_1 \geq \exp(N_2^{1/2+\varepsilon})$, or equivalently, when $N_2 \leq (\log N_1)^{2-\varepsilon}$. Positive density results were also conjectured in [1] and proven in part in [2].

In this paper we examine analogous questions for abelian surfaces over finite fields. Here and throughout, q will denote the prime power p^r , and A will denote an abelian surface over the finite field \mathbb{F}_q . The points on A over \mathbb{F}_q possess the structure of a finite abelian group $A(\mathbb{F}_q)$ of rank at most 4, i.e.,

$$A(\mathbb{F}_q) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_1n_2\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3\mathbb{Z} \times \mathbb{Z}/n_1n_2n_3n_4\mathbb{Z} \tag{1.2}$$

for some positive integers n_1, n_2, n_3, n_4 . For the sake of convenience, we will use the notation $G(n_1, n_2, n_3, n_4)$ to refer to the group on the right hand side of (1.2). We then want to study which of the groups $G(n_1, n_2, n_3, n_4)$ actually occur when we vary over all finite fields \mathbb{F}_q and over all abelian surfaces A/\mathbb{F}_q .

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For fixed q , a characterization of the groups occurring as the group of points on a general abelian variety was recently found by Rybakov [8, 9]. Rybakov’s elegant criterion relates the Newton polygon of the characteristic polynomial of the variety to the Hodge polygon of the group. The work of Rybakov may be viewed as generalizing Rück’s characterization for elliptic curves [7] to abelian varieties of any dimension. We give a detailed description of these results in Section 2.

As with the case of elliptic curves, we expect that the “very split” groups $G(n_1, n_2, n_3, n_4)$ (viz., when n_1, n_2 are large with respect to n_3, n_4) are less likely to occur. This is compatible with the general philosophy of the Cohen-Lenstra heuristics, which predict that random abelian groups naturally occur with probability inversely proportional to the size of their automorphism groups. Note that the very split groups have many more automorphisms than the cyclic group of the same size. In fact, Rybakov’s criterion shows that whenever there is an abelian variety with N points over \mathbb{F}_q , the cyclic group of order N will always occur.

We now state our main results. We recall that an abelian variety is simple if it is not isogenous to a product of abelian varieties of lower dimension. Our first result is that some groups never occur for simple abelian surfaces over \mathbb{F}_q . In particular, when n_1 is too large with respect to n_2, n_3, n_4 , the group $G(n_1, n_2, n_3, n_4)$ does not arise as the group of points on any simple abelian variety over any finite field. This is different from the case of elliptic curves, where such a statement is true only in a probabilistic sense, viz., very split groups occur with density zero as proven in [2].

Theorem 1.1. *Suppose that n_1, n_2, n_3, n_4 are positive integers. If*

$$n_1 \geq 60n_2^{1/4} n_3^{3/2} n_4^{3/4} + 1,$$

then for every q , there is no simple abelian surface A/\mathbb{F}_q with $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$.

We also show that fewer groups occur in a probabilistic sense. Our next result essentially says that if n_1 or n_2 is very large compared to n_3 and n_4 , then $G(n_1, n_2, n_3, n_4)$ occurs with probability zero. Given $N_1, N_2, N_3, N_4 \geq 1$, we define $S(N_1, N_2, N_3, N_4)$ to be the set of quadruples (n_1, n_2, n_3, n_4) for which $N_j \leq n_j \leq 2N_j$ for $1 \leq j \leq 4$ and there exists a prime power q and a simple abelian surface A/\mathbb{F}_q with $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$. Throughout, we write $f = \underline{o}(g)$ as $x \rightarrow \infty$ if $f/g \rightarrow 0$ as $x \rightarrow \infty$.

Theorem 1.2. *If*

$$\frac{N_1 N_2^{1/4}}{N_3^{1/2} N_4^{1/4}} \rightarrow \infty$$

as $N_2 N_4 \rightarrow \infty$, then

$$\#S(N_1, N_2, N_3, N_4) = \underline{o}(N_1 N_2 N_3 N_4)$$

as $N_2 N_4 \rightarrow \infty$.

2. WEIL POLYNOMIALS AND GROUPS OF ABELIAN SURFACES

A classification of simple abelian varieties over \mathbb{F}_q (up to \mathbb{F}_q -isogeny) is given by Tate-Honda theory, which gives a one-to-one correspondence between isogeny classes of simple abelian varieties over \mathbb{F}_q and conjugacy classes of Weil numbers (algebraic integers whose conjugates have absolute value $q^{1/2}$). This classification can be stated using the characteristic polynomial of the Frobenius endomorphism π_A of A/\mathbb{F}_q . This polynomial, which we denote by $f_A(T)$, determines A up to isogeny, and it has Weil numbers as its roots. For an abelian surface A/\mathbb{F}_q , we write

$$f_A(T) = T^4 + a_1 T^3 + a_2 T^2 + a_1 q T + q^2.$$

The number of \mathbb{F}_q -rational points on A is equal to $f_A(1)$ and hence is an invariant of the isogeny class. The fact that the roots of $f_A(T)$ are Weil numbers implies that

$$(\sqrt{q} - 1)^4 \leq \#A(\mathbb{F}_q) \leq (\sqrt{q} + 1)^4. \quad (2.1)$$

If A is a simple abelian surface, then $f_A(T) = h_A(T)^e$ where $h_A(T)$ is an irreducible polynomial in $\mathbb{Z}[T]$ whose roots are Weil numbers. Furthermore, the endomorphism algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field if and only if $e = 1$. Computing the local invariants of the algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ allows one to obtain a correspondence between the set of simple abelian surfaces over \mathbb{F}_q such that $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field and the set of irreducible polynomials $f(T)$ of degree 4 whose roots are Weil numbers and whose monic irreducible divisors $f_i(T)$ over \mathbb{Q}_p have integer values of $\nu_p(f_i(0))/\nu_p(q)$. Here and throughout, we use the notation ν_p to denote the usual p -adic valuation. Rück [6] gave the following explicit characterization of these polynomials.

Theorem 2.1 (Rück). *The set of $f_A(T)$ for all abelian varieties A over \mathbb{F}_q of dimension 2 whose algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field is equal to the set of polynomials $f(T) = T^4 + a_1T^3 + a_2T^2 + a_1qT + q^2$ where the integers a_1 and a_2 satisfy the conditions*

- (a.) $|a_1| < 4q^{1/2}$, $2|a_1|q^{1/2} - 2q < a_2 < a_1^2/4 + 2q$,
- (b.) $a_1^2 - 4a_2 + 8q$ is not a square in \mathbb{Z} , and
- (c.) either
 - (i.) $\nu_p(a_1) = 0$, $\nu_p(a_2) \geq r/2$ and $(a_2 + 2q)^2 - 4qa_1^2$ is not a square in \mathbb{Z}_p ,
 - (ii.) $\nu_p(a_2) = 0$, or
 - (iii.) $\nu_p(a_1) \geq r/2$, $\nu_p(a_2) \geq r$, and $f(T)$ has no root in \mathbb{Z}_p .

The polynomials $f_A(T)$ corresponding to simple abelian surfaces A over \mathbb{F}_q whose algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is not a field are much rarer. They can be described explicitly as well.

Theorem 2.2 (Waterhouse, Xing). *The characteristic polynomial $f_A(T)$ of any simple abelian variety A of dimension 2 over \mathbb{F}_q whose algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is not field must be of the form*

- (a.) $f_A(T) = (T^2 - q)^2$ and r is odd,
- (b.) $f_A(T) = (T^2 + q)^2$, r is even, and $p \equiv 1 \pmod{4}$, or
- (c.) $f_A(T) = (T^2 \pm q^{1/2}T + q)^2$, r is even, and $p \equiv 1 \pmod{3}$.

The group structures for these “exceptional” polynomials $f_A(T)$ were studied by Xing in [12] and [13]. In the respective cases (corresponding to Theorem 2.2), Xing showed that the group structures which arise are precisely

- (a.) $(\mathbb{Z}/(q-1)\mathbb{Z})^2$, $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/\frac{q-1}{2}\mathbb{Z})^2$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\frac{q-1}{2}\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$;
- (b.) $(\mathbb{Z}/(q+1)\mathbb{Z})^2$; or
- (c.) $(\mathbb{Z}/(q \pm q^{1/2} + 1)\mathbb{Z})^2$.

We refer the reader to [12] for a precise description of when each group corresponding to the first case arises. Thus, the abelian surfaces A whose algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is not a field give rise to very few groups $G(n_1, n_2, n_3, n_4)$. More importantly, $n_1, n_2 \leq 2$ for all such groups, and hence they do not satisfy the conditions of Theorem 1.1 or Theorem 1.2. Therefore, we exclude this case from consideration for the remainder of the paper.

For the typical case of abelian surfaces whose algebra is a field, there is a very elegant criterion due to Rybakov [8, 9] that characterizes those isogeny classes which contain a variety A with $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$. The result of Rybakov applies to abelian varieties of any dimension $g \geq 1$. We state it below in full generality and then for the particular case of abelian surfaces. We first need some definitions.

Let ℓ be a prime, and let $Q(T) = \sum_i Q_i T^i$ be a polynomial of degree d with $Q(0) = Q_0 \neq 0$. The *Newton polygon* $\text{Np}_\ell(Q)$ is the boundary (without vertical lines) of the lower convex hull of the points $(i, \nu_\ell(Q_i))$ for $0 \leq i \leq d$ in \mathbb{R}^2 . Now let $0 \leq m_1 \leq m_2 \leq \dots \leq m_r$ be nonnegative integers, and let $H = \bigoplus_{i=1}^r \mathbb{Z}/\ell^{m_i} \mathbb{Z}$. The *Hodge polygon* $\text{Hp}_\ell(H, r)$ is the convex polygon with vertices $(i, \sum_{j=1}^{r-i} m_j)$ for $0 \leq i < r$. Given an abelian group G , we let G_ℓ denote the ℓ -primary component of G . The following is the main result of [8].

Theorem 2.3 (Rybakov). *Let A be an abelian variety of dimension g over a finite field whose algebra $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field. Let $f_A(T)$ denote its characteristic polynomial, and let G be an abelian group of order $f_A(1)$ that can be generated by $2g$ or fewer elements. Then G is the group of points on some variety in the isogeny class of A if and only if $\text{Np}_\ell(f_A(1 - T))$ lies on or above $\text{Hp}_\ell(G_\ell, 2g)$ for every prime number ℓ .*

For the case of abelian surfaces, we rewrite the conditions of Theorem 2.3 explicitly as follows.

Corollary 2.4. *Let A/\mathbb{F}_q be an abelian surface, and suppose that $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field. Let $f_A(T) = T^4 + a_1 T^3 + a_2 T^2 + a_1 q T + q^2$ denote its Weil polynomial. Then the isogeny class of A contains a variety with group of points isomorphic to $G(n_1, n_2, n_3, n_4)$ if and only if*

$$n_1^4 n_2^3 n_3^2 n_4 = f_A(1) = q^2 + a_1 q + a_2 + a_1 + 1 \quad (2.2)$$

and

$$4 + 3a_1 + 2a_2 + qa_1 \equiv 0 \pmod{n_1^3 n_2^2 n_3}, \quad (2.3)$$

$$6 + 3a_1 + a_2 \equiv 0 \pmod{n_1^2 n_2}, \quad (2.4)$$

$$4 + a_1 \equiv 0 \pmod{n_1}. \quad (2.5)$$

We remark that Corollary 2.4 implies that if $f_A(1) = N$, then the cyclic group of order N occurs as a group of points on some abelian surface in the isogeny class of A since in that case we have $n_1 = n_2 = n_3 = 1$, so the congruences (2.3) – (2.5) are trivially satisfied.

3. KEY PROPOSITION

In this section we prove the following key proposition. As is somewhat common, for any real number x , we write $\|x\|$ for the distance between x and its nearest integer neighbor. To ease notation, we define

$$\delta = \delta(n_1, n_2, n_3, n_4) = \begin{cases} 1 & \text{if } 2n_3\sqrt{n_2 n_4} \in \mathbb{Z}, \\ \|2n_3\sqrt{n_2 n_4}\| & \text{otherwise} \end{cases} \quad (3.1)$$

for any positive integers n_1, n_2, n_3, n_4 .

Proposition 3.1. *Suppose that A/\mathbb{F}_q is a simple abelian surface and that $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field. Suppose further that $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$. Then*

$$n_1 < \frac{10n_3^{1/2} n_4^{1/4}}{\delta n_2^{1/4}} + \frac{1}{n_2^{3/4} n_3^{1/2} n_4^{1/4}}.$$

Theorems 1.1 and 1.2 will follow from Proposition 3.1. Theorem 1.1 follows by specifying lower bounds for $\|\sqrt{m}\|$ that are valid for every integer m . For the details, see Section 4. Theorem 1.2 follows from the fact that the triple sequence $2n_3\sqrt{n_2 n_4}$ is uniformly distributed modulo one; we will prove this in Section 5.

First, we use the congruences of Corollary 2.4 to derive a simpler congruence on a_1 .

Lemma 3.2. *Suppose that $q, a_1, a_2, n_1, n_2, n_3,$ and n_4 satisfy (2.2)–(2.5). Then*

$$a_1 \equiv -2(q+1) \pmod{n_1^2 n_2}.$$

Proof. Reducing (2.2) and (2.3) modulo $n_1^3 n_2^2$ yields

$$q^2 + a_1 q + a_2 + a_1 + 1 \equiv 0 \pmod{n_1^3 n_2^2}, \quad (3.2)$$

$$4 + 3a_1 + 2a_2 + qa_1 \equiv 0 \pmod{n_1^3 n_2^2}. \quad (3.3)$$

Reducing the above congruences modulo $n_1^2 n_2$ and taking their difference gives

$$2a_1 + a_2 + 3 - q^2 \equiv 0 \pmod{n_1^2 n_2}.$$

Subtracting this from (2.4), we obtain

$$a_1 + 3 + q^2 \equiv 0 \pmod{n_1^2 n_2}. \quad (3.4)$$

The remainder of the proof is devoted to showing that $n_1^2 n_2 \mid (q-1)^2$, which together with (3.4) implies the desired congruence $a_1 \equiv -2(q+1) \pmod{n_1^2 n_2}$.

Taking twice (3.2) and subtracting off (3.3) yields

$$2q^2 + a_1 q - a_1 - 2 \equiv 0 \pmod{n_1^3 n_2^2}. \quad (3.5)$$

From (3.4) we know that there is an integer k such that $a_1 = -3 - q^2 + kn_1^2 n_2$. After some slight rearrangement, plugging this expression for a_1 into (3.5) gives

$$kn_1^2 n_2 (q-1) - (q-1)^3 \equiv 0 \pmod{n_1^3 n_2^2}. \quad (3.6)$$

Working prime by prime, we will show that (3.6) implies that $n_1^2 n_2 \mid (q-1)^2$. To this end let ℓ be an arbitrary prime, and suppose that $\nu_\ell(n_1^2 n_2) = r$. Then we want to show that $\nu_\ell((q-1)^2) \geq r$. Assume for the sake of contradiction that $\nu_\ell((q-1)^2) < r$. Since $\nu_\ell(kn_1^2 n_2) \geq r$, it follows that $\nu_\ell(kn_1^2 n_2 - (q-1)^3) = \nu_\ell((q-1)^2)$, and hence

$$\nu_\ell(kn_1^2 n_2 (q-1) - (q-1)^3) = \nu_\ell(q-1) + \nu_\ell(kn_1^2 n_2 - (q-1)^2) = 3\nu_\ell(q-1) < \frac{3r}{2}.$$

On the other hand since $n_1^3 n_2^2$ divides $kn_1^2 n_2 (q-1) - (q-1)^3$, it follows that

$$3\nu_\ell(n_1) + 2\nu_\ell(n_2) \leq \nu_\ell(kn_1^2 n_2 (q-1) - (q-1)^3) < \frac{3r}{2} = \frac{3}{2}(2\nu_\ell(n_1) + \nu_\ell(n_2)).$$

However, this implies that $\nu_\ell(n_2) < 0$, which is impossible since n_2 is an integer. \square

Lemma 3.3. *Suppose that A/\mathbb{F}_q is a simple abelian surface, $\text{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$ is a field, and $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$. If $f_A(T) = T^4 + a_1 T^3 + a_2 T^2 + qa_1 T + q^2$ is the characteristic polynomial of A/\mathbb{F}_q , then there exists an integer k such that*

$$a_1 = kn_1^2 n_2 - 2(q+1)$$

and

$$2n_3 \sqrt{n_2 n_4} \left(\frac{\sqrt{q}-1}{\sqrt{q}+1} \right)^2 < k < 2n_3 \sqrt{n_2 n_4} \left(\frac{\sqrt{q}+1}{\sqrt{q}-1} \right)^2. \quad (3.7)$$

Proof. By Theorem 2.1(a), we know that $-4\sqrt{q} < a_1 < 4\sqrt{q}$. By Lemma 3.2, there exists an integer k such that $a_1 = n_1^2 n_2 k - 2(q+1)$. Substituting this into the bounds for a_1 and adding $2(q+1)$ to each side of the inequalities yields

$$2q - 4\sqrt{q} + 2 < n_1^2 n_2 k < 2q + 4\sqrt{q} + 2.$$

Factoring and dividing through by $n_1^2 n_2$ allows us to obtain

$$\frac{2(\sqrt{q}-1)^2}{n_1^2 n_2} < k < \frac{2(\sqrt{q}+1)^2}{n_1^2 n_2}.$$

Since $\#A(\mathbb{F}_q) = n_1^4 n_2^3 n_3^2 n_4$, the Weil bound (2.1) implies that

$$\frac{(\sqrt{q} - 1)^2}{n_3 \sqrt{n_2 n_4}} \leq n_1^2 n_2 \leq \frac{(\sqrt{q} + 1)^2}{n_3 \sqrt{n_2 n_4}}.$$

Together these bounds imply (3.7). \square

For q large enough, the interval from Lemma 3.3 will contain at most one integer k . The following lemma makes this statement precise. Recall the definition of δ given by (3.1).

Lemma 3.4. *If $\sqrt{q} \geq 10n_3\sqrt{n_2 n_4}/\delta$, then the interval (3.7) contains no integral values of k unless $2n_3\sqrt{n_2 n_4}$ is an integer, in which case $k = 2n_3\sqrt{n_2 n_4}$.*

Proof. To further ease notation, let $m = 2n_3\sqrt{n_2 n_4}$. Note that the interval $(m - \delta, m + \delta)$ does not contain an integer unless $m = 2n_3\sqrt{n_2 n_4}$ is itself an integer, in which case it is the only such integer. Since

$$\left(\frac{\sqrt{q} + 1}{\sqrt{q} - 1}\right)^2 = 1 + \frac{4\sqrt{q}}{(\sqrt{q} - 1)^2} \quad \text{and} \quad \left(\frac{\sqrt{q} - 1}{\sqrt{q} + 1}\right)^2 = 1 - \frac{4\sqrt{q}}{(\sqrt{q} + 1)^2},$$

it follows that the interval (3.7) is contained in the interval $(m - \delta, m + \delta)$ if and only if

$$m \frac{4\sqrt{q}}{(\sqrt{q} - 1)^2} \leq \delta.$$

Factoring the latter inequality and dividing by δ yields

$$0 \leq \left(\sqrt{q} - \frac{2m + \delta - 2\sqrt{m^2 + m\delta}}{\delta}\right) \left(\sqrt{q} - \frac{2m + \delta + 2\sqrt{m^2 + m\delta}}{\delta}\right). \quad (3.8)$$

Now, since

$$\frac{2m + \delta - 2\sqrt{m^2 + m\delta}}{\delta} \leq 1,$$

it follows that (3.8) holds if and only if

$$\sqrt{q} \geq \frac{2m + \delta + 2\sqrt{m^2 + m\delta}}{\delta}.$$

However,

$$\frac{2m + \delta + 2\sqrt{m^2 + m\delta}}{\delta} \leq \frac{2m + 1 + 2\sqrt{m^2 + m + 1/4}}{\delta} = \frac{4m + 2}{\delta} \leq \frac{5m}{\delta},$$

and so (3.8) holds if $\sqrt{q} \geq 5m/\delta = 10n_3\sqrt{n_2 n_4}/\delta$. \square

Proof of Proposition 3.1. First, suppose that $k = 2n_3\sqrt{n_2 n_4}$ is an integer and

$$a_1 = kn_1^2 n_2 - 2(q + 1) = 2n_1^2 n_2^{3/2} n_3 n_4^{1/2} - 2(q + 1).$$

Then substitution into (2.2) gives

$$a_2 = n_1^4 n_2^3 n_3^2 n_4 - 1 - \left(2n_1^2 n_2^{3/2} n_3 n_4^{1/2} - 2(q + 1)\right) (q + 1) - q^2.$$

Under these assumptions, we then find that

$$a_1^2 - 4a_2 + 8q = 0.$$

According to Theorem 2.1(b), this contradicts the assumption that $\text{End}_{\mathbb{F}_q} \otimes \mathbb{Q}$ is field. Therefore, by Lemmas 3.3 and 3.4, regardless of whether $2n_3\sqrt{n_2 n_4}$ is an integer, we see

that if $A(\mathbb{F}_q) \simeq G(n_1, n_2, n_3, n_4)$ and $\text{End}_{\mathbb{F}_q} \otimes \mathbb{Q}$ is a field, then $\sqrt{q} < 10n_3\sqrt{n_2n_4}/\delta$. Using this together with the Weil bound (2.1), we have that

$$n_1 n_2^{3/4} n_3^{2/4} n_4^{1/4} \leq \sqrt{q} + 1 < 10n_3\sqrt{n_2n_4}/\delta + 1.$$

Whence,

$$n_1 < \frac{10n_3^{1/2} n_4^{1/4}}{\delta n_2^{1/4}} + \frac{1}{n_2^{3/4} n_3^{1/2} n_4^{1/4}}.$$

□

4. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 can be deduced from the following simple observation, which gives a lower bound for $\|2n_3\sqrt{n_2n_4}\|$. As usual, for any real number x , we write $[x]$ for the largest integer less than or equal to x , and $\{x\} = x - [x]$ for the fractional part of x .

Lemma 4.1. *Let m be an integer that is not a perfect square. Then*

$$\|\sqrt{m}\| > \frac{1}{3\sqrt{m}}.$$

Proof. Since $\sqrt{m} = [\sqrt{m}] + \{\sqrt{m}\}$, upon squaring both sides, we find that

$$\begin{aligned} m &= [\sqrt{m}]^2 + 2\{\sqrt{m}\}[\sqrt{m}] + \{\sqrt{m}\}^2 \\ &= [\sqrt{m}]^2 + (\sqrt{m} + [\sqrt{m}])\{\sqrt{m}\} \\ &\leq [\sqrt{m}]^2 + 2\sqrt{m}\{\sqrt{m}\}. \end{aligned}$$

Therefore, since $1 < m - [\sqrt{m}]^2$, we have $\{\sqrt{m}\} > 1/2\sqrt{m}$.

Similarly, since we can write $\sqrt{m} = [\sqrt{m}] + 1 - (1 - \{\sqrt{m}\})$, we have that

$$\begin{aligned} m &= ([\sqrt{m}] + 1)^2 - 2(1 - \{\sqrt{m}\})([\sqrt{m}] + 1) + (1 - \{\sqrt{m}\})^2 \\ &= ([\sqrt{m}] + 1)^2 - (\sqrt{m} + [\sqrt{m}] + 1)(1 - \{\sqrt{m}\}) \\ &\geq ([\sqrt{m}] + 1)^2 - (2\sqrt{m} + 1)(1 - \{\sqrt{m}\}). \end{aligned}$$

Therefore, since $1 \leq ([\sqrt{m}] + 1)^2 - m$, we obtain $1 - \{\sqrt{m}\} \geq 1/(2\sqrt{m} + 1) > 1/3\sqrt{m}$. □

Proof of Theorem 1.1. By Proposition 3.1, if there is a prime power q and a simple abelian surface A/\mathbb{F}_q with group $G(n_1, n_2, n_3, n_4)$, then

$$n_1 < \frac{10n_3^{1/2} n_4^{1/4}}{\delta n_2^{1/4}} + \frac{1}{n_2^{3/4} n_3^{1/2} n_4^{1/4}} \leq \frac{10n_3^{1/2} n_4^{1/4}}{\delta n_2^{1/4}} + 1,$$

where δ is as defined by (3.1). By Lemma 4.1,

$$n_1 < 60n_2^{1/4} n_3^{3/2} n_4^{3/4} + 1$$

since $\delta \geq (6n_3\sqrt{n_2n_4})^{-1}$. □

5. PROOF OF THEOREM 1.2

In this section we use the standard notation $f \ll g$ to mean that there exists a positive constant c such that $|f| \leq cg$. We also use the notation $n \asymp N$ (in a somewhat nonstandard way) as a shorthand for $N \leq n \leq 2N$.

To prove Theorem 1.2, we use the fact that for most triples of integers (n_2, n_3, n_4) with $n_j \asymp N_j$ ($2 \leq j \leq 4$), the distance between $2n_3\sqrt{n_2n_4}$ and the nearest integer is larger than any function tending to zero as $N_2N_4 \rightarrow \infty$. This follows from the uniform distribution of $2n_3\sqrt{n_2n_4}$ modulo one; see Theorem 5.6 below. For the sake of completeness, we review much of the relevant material here.

Let

$$\mathcal{T}(N_2, N_3, N_4) = \{(n_2, n_3, n_4) : n_2 \asymp N_2, n_3 \asymp N_3, n_4 \asymp N_4\},$$

and let $\{f(n_2, n_3, n_4) : n_2, n_3, n_4 \geq 1\}$ be any triply indexed sequence of real numbers. For $0 \leq \alpha < \beta \leq 1$, let

$$Z_f(N_2, N_3, N_4; \alpha, \beta) = \#\{(n_2, n_3, n_4) \in \mathcal{T}(N_2, N_3, N_4) : \alpha \leq \{f(n_2, n_3, n_4)\} \leq \beta\},$$

where, as in the previous section, $\{f(n_2, n_3, n_4)\}$ denotes the fractional part of $f(n_2, n_3, n_4)$. We say that the sequence $f(n_2, n_3, n_4)$ is uniformly distributed modulo one if

$$\lim_{N_2, N_3, N_4 \rightarrow \infty} \frac{Z_f(N_2, N_3, N_4; \alpha, \beta)}{N_2N_3N_4} = \beta - \alpha.$$

By Weyl's criterion, this is equivalent to showing that

$$E_k(N_2, N_3, N_4) := \sum_{\substack{n_2 \asymp N_2, \\ n_3 \asymp N_3, \\ n_4 \asymp N_4}} e(kf(n_2, n_3, n_4)) = o(N_2N_3N_4)$$

for every integer $k \neq 0$. As usual, we have written $e(x) = e^{2\pi ix}$. We can put this equivalence in quantitative form using the Selberg polynomials. This is explained in Chapter 1 of [5] for a sequence of one variable. The proof for a sequence of three variables $f(n_2, n_3, n_4)$ follows along the same lines. The next theorem is then the analogue of [5, Chapter 1, Theorem 1] for triple sequences.

Theorem 5.1. *Let $f(n_2, n_3, n_4)$ be a sequence of real numbers, and let $0 \leq \alpha \leq \beta \leq 1$. Then*

$$\begin{aligned} & |Z_f(N_2, N_3, N_4; \alpha, \beta) - (\beta - \alpha) \#\mathcal{T}(N_2, N_3, N_4)| \\ & \leq \frac{\#\mathcal{T}(N_2, N_3, N_4)}{K+1} + 2 \sum_{k=1}^K \left(\frac{1}{K+1} + \min\left(\beta - \alpha, \frac{1}{\pi k}\right) \right) |E_k(N_2, N_3, N_4)| \end{aligned} \quad (5.1)$$

for any positive integers N_2, N_3, N_4 , and K .

Proof. For each positive integer K , let

$$S_K^+(n) = \sum_{-K \leq k \leq K} \widehat{S}_K^+(k) e(kn)$$

be the Selberg polynomial upper bounding the characteristic function of $[\alpha, \beta]$ as defined in [5, p. 6]. Then

$$\begin{aligned} Z_f(N_2, N_3, N_4; \alpha, \beta) &\leq \sum_{\substack{n_2 \asymp N_2 \\ n_3 \asymp N_3 \\ n_4 \asymp N_4}} S_K^+(f(n_2, n_3, n_4)) \\ &= \sum_{-K \leq k \leq K} \widehat{S}_K^+(k) \sum_{\substack{n_2 \asymp N_2 \\ n_3 \asymp N_3 \\ n_4 \asymp N_4}} e(kf(n_2, n_3, n_4)). \end{aligned}$$

Now, since

$$\widehat{S}_K^+(0) = \beta - \alpha + \frac{1}{K+1}$$

and

$$E_0(N_2, N_3, N_4) = \#\mathcal{T}(N_2, N_3, N_4),$$

we have that

$$\begin{aligned} Z_f(N_2, N_3, N_4; \alpha, \beta) - (\beta - \alpha) \#\mathcal{T}(N_2, N_3, N_4) \\ \leq \frac{\#\mathcal{T}(N_2, N_3, N_4)}{K+1} + \sum_{\substack{-K \leq k \leq K \\ k \neq 0}} \widehat{S}_K^+(k) E_k(N_2, N_3, N_4). \end{aligned}$$

It follows from properties of Selberg polynomials that

$$|\widehat{S}_K^+(k)| \leq \frac{1}{K+1} + \min\left(\beta - \alpha, \frac{1}{\pi|k|}\right)$$

for $0 < |k| \leq K$. See [5, p. 8] for example. Combining the inequalities from above, we have

$$\begin{aligned} Z_f(N_2, N_3, N_4; \alpha, \beta) - (\beta - \alpha) \#\mathcal{T}(N_2, N_3, N_4) \\ \leq \frac{\#\mathcal{T}(N_2, N_3, N_4)}{K+1} + 2 \sum_{1 \leq k \leq K} \left(\frac{1}{K+1} + \min\left(\beta - \alpha, \frac{1}{\pi|k|}\right) \right) |E_k(N_2, N_3, N_4)|. \end{aligned}$$

Using the Selberg polynomials $S_K^-(n)$ as defined in [5, p. 6], the other inequality follows, as does the theorem. \square

For the remainder of the paper, we will specialize to the the sequence $f(n_2, n_3, n_4) = 2n_3\sqrt{n_2n_4}$. We now bound the sum appearing in Theorem 5.1 to show that the sequence $2n_3\sqrt{n_2n_4}$ is uniformly distributed modulo one. In order to obtain our result without any conditions on the relative sizes of the parameters N_2, N_3, N_4 , we bound the sum appearing in (5.1) in two different ways (Lemmas 5.3 and 5.5). First, we use the following result from [4, p. 77].

Lemma 5.2. *Let $g(t)$ be a real, continuously differentiable function on the interval $[a, b]$, with $|g'(t)| \geq \lambda > 0$, and let $N > 0$. Then*

$$\sum_{a \leq n \leq b} \min\{N, 1/\|g(n)\|\} \ll (|g(b) - g(a)| + 1) \left(N + \frac{1}{\lambda} \log(b - a + 2) \right).$$

Lemma 5.3. *For every $\varepsilon > 0$ and $K \geq 1$,*

$$\sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| \ll (N_2 N_4)^{1/2+\varepsilon} N_3 K + (N_2 N_4)^{1+\varepsilon} \log 2K.$$

Proof. Let

$$b_n = \sum_{n_2 \asymp N_2} \sum_{\substack{n_4 \asymp N_4 \\ n_2 n_4 = n}} 1,$$

and note that $b_n \ll n^{\varepsilon/2}$. Recall the well-known bound

$$\sum_{n \asymp N} e(\alpha n) \ll \min \{N, 1/|\alpha|\}.$$

See [3, p. 199] for example. Applying Lemma 5.2, we have

$$\begin{aligned} \sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| &= \sum_{k \leq K} \frac{1}{k} \sum_{N_2 N_4 \leq n \leq 4N_2 N_4} b_n \sum_{n_3 \asymp N_3} e(2kn^{1/2}n_3) \\ &\ll (N_2 N_4)^{\varepsilon/2} \sum_{k \leq K} \frac{1}{k} \sum_{N_2 N_4 \leq n \leq 4N_2 N_4} \min \left\{ N_3, \frac{1}{\|2kn^{1/2}\|} \right\} \\ &\ll (N_2 N_4)^{\varepsilon/2} \sum_{k \leq K} (N_2 N_4)^{1/2} (N_3 + k^{-1}(N_2 N_4)^{1/2} \log(2N_2 N_4)) \\ &\ll (N_2 N_4)^{\varepsilon} ((N_2 N_4)^{1/2} N_3 K + (N_2 N_4) \log 2K). \end{aligned}$$

□

We now bound the same sum using the following consequence of the van der Corput method found in [10, p. 94].

Lemma 5.4. *Let $g(t)$ be a twice continuously differentiable function on the interval $[a, b]$ such that $|g''(t)| \asymp \lambda > 0$. Then*

$$\sum_{a \leq n \leq b} e(g(n)) \ll (b - a + 1)\lambda^{1/2} + \lambda^{-1/2}.$$

Lemma 5.5. *For every $K \geq 1$,*

$$\sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| \ll K^{1/2} N_3^{3/2} (N_2 N_4)^{3/4} + N_3^{1/2} (N_2 N_4)^{3/4}.$$

Proof. We first apply Lemma 5.4 with $g(t) = 2kn_3\sqrt{n_2 t}$, noting that $|g''(t)| = kn_3\sqrt{n_2}/2t^{3/2}$. This yields

$$\begin{aligned} \sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| &\ll \sum_{k \leq K} \frac{1}{k} \sum_{\substack{n_2 \asymp N_2, \\ n_3 \asymp N_3}} \left(N_4^{1/4} (kn_3\sqrt{n_2})^{1/2} + N_4^{3/4} (kn_3\sqrt{n_2})^{-1/2} \right) \\ &\ll N_4^{1/4} K^{1/2} N_3^{3/2} N_2^{5/4} + N_4^{3/4} N_3^{1/2} N_2^{3/4}. \end{aligned}$$

Then, applying Lemma 5.4 again with $g(t) = 2kn_3\sqrt{n_4 t}$, we see that the same bound holds with the roles of N_2 and N_4 reversed. Therefore, we have

$$\begin{aligned} \sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| &\ll K^{1/2} N_3^{3/2} \min\{N_2^{5/4} N_4^{1/4}, N_2^{1/4} N_4^{5/4}\} + N_3^{1/2} (N_2 N_4)^{3/4} \\ &\ll K^{1/2} N_3^{3/2} (N_2 N_4)^{3/4} + N_3^{1/2} (N_2 N_4)^{3/4}. \end{aligned}$$

□

Combining Lemmas 5.3 and 5.5, we now show that the triple sequence $2n_3\sqrt{n_2 n_4}$ is uniformly distributed modulo one.

Theorem 5.6. *Let $N_2, N_3, N_4 \geq 1$, and $0 \leq \alpha < \beta \leq 1$. Then*

$$\lim_{N_2 N_4 \rightarrow \infty} \frac{Z_f(N_2, N_3, N_4; \alpha, \beta)}{N_2 N_3 N_4} = \beta - \alpha.$$

Remark. Note that we do not require that each of N_2, N_3 and N_4 tends to infinity in the above limit. Rather, we only require that the product $N_2 N_4 \rightarrow \infty$.

Proof. Fix $0 < \varepsilon < 1/16$. Applying Lemmas 5.3 and 5.5 with $K = (N_2 N_4)^{1/4}$, we see that

$$\begin{aligned} \sum_{k \leq K} \frac{1}{k} |E_k(N_2, N_3, N_4)| &\ll (N_2 N_4)^{3/4+\varepsilon} N_3 + \min\{(N_2 N_4)^{1+\varepsilon}, N_3^{3/2} (N_2 N_4)^{7/8}\} \\ &\ll (N_2 N_4)^{3/4+\varepsilon} N_3 + (N_2 N_4)^{15/16+\varepsilon} N_3^{3/4}. \end{aligned}$$

Since (with this same choice of K) we have $N_2 N_3 N_4 / K = (N_2 N_4)^{3/4} N_3$, using the above bound in Theorem 5.1 yields the theorem. \square

Proof of Theorem 1.2. Let $F(N_2, N_4)$ be any function tending to infinity with $N_2 N_4$ and satisfying the bound

$$F(N_2, N_4) \leq \frac{N_1 N_2^{1/4}}{18 N_3^{1/2} N_4^{1/4}}. \quad (5.2)$$

Without loss of generality, we may assume that $N_2 N_4$ is large enough that $F(N_2, N_4) \geq 1$. Hence, we may write

$$\#S(N_1, N_2, N_3, N_4) = \#S_1(N_1, N_2, N_3, N_4) + \#S_2(N_1, N_2, N_3, N_4),$$

where

$$\begin{aligned} S_1(N_1, N_2, N_3, N_4) &:= \{(n_1, n_2, n_3, n_4) \in S(N_1, N_2, N_3, N_4) : \|2n_3 \sqrt{n_2 n_4}\| \leq 1/F(N_2, N_4)\}, \\ S_2(N_1, N_2, N_3, N_4) &:= \{(n_1, n_2, n_3, n_4) \in S(N_1, N_2, N_3, N_4) : \|2n_3 \sqrt{n_2 n_4}\| > 1/F(N_2, N_4)\}. \end{aligned}$$

It follows from Theorem 5.6 that $\#S_1(N_1, N_2, N_3, N_4) = \underline{o}(N_1 N_2 N_3 N_4)$ as $N_2 N_4 \rightarrow \infty$. On the other hand, if $(n_1, n_2, n_3, n_4) \in S_2(N_1, N_2, N_3, N_4)$, then by Proposition 3.1

$$\begin{aligned} N_1 \leq n_1 &< \frac{10 n_3^{1/2} n_4^{1/4}}{\|2n_3 \sqrt{n_2 n_4}\| n_2^{1/4}} + \frac{1}{n_2^{3/4} n_3^{1/2} n_4^{1/4}} \\ &< \frac{10(2N_3)^{1/2} (2N_4)^{1/4}}{1/F(N_2, N_4) N_2^{1/4}} + \frac{1}{N_2^{3/4} N_3^{1/2} N_4^{1/4}} \\ &< 18F(N_2, N_4) \frac{N_3^{1/2} N_4^{1/4}}{N_2^{1/4}}. \end{aligned}$$

However, this contradicts our choice of $F(N_2, N_4)$ that satisfies (5.2). Therefore, we conclude that $S_2(N_1, N_2, N_3, N_4)$ is empty, and hence $S(N_1, N_2, N_3, N_4) = \underline{o}(N_1 N_2 N_3 N_4)$ as $N_2 N_4 \rightarrow \infty$. \square

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